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TRANSLATION

PROBLEMS OF THE STATISTICAL THEORY OF RADAR

By

P. A. Bakut, I. A. Bol'shakov, et al.

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

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FOREWORD

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EDITED MACHINE TRANSLATION

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BY: P. A. Bakut, I. A. Bol'shakov, et al.

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VOPROSY STATISTICHESKOY TEORII RADIOLOKATSII

pod obshchey redaktsiyey
professora G. P. Tartakovskogo

Tom I

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TABLE OF CONTENTS

ANNOTATION

This book is a monograph on the statistical theory of radar. It consists of two volumes. In the first volume the basic problems of the theory of detection are expounded, in the second -- the problems of radar measurements and several problems on target resolution. Methods for analysis and synthesis of radar systems are treated, as are many results and rules obtained by these methods. The book is designed for teachers and those acquainted with the basic aspects of the theories of probability and the theories of random processes. The necessary information from the theory of statistical resolution is given in the book.

The book is intended for scientific workers and engineers, those concerned with radar problems, and also for post-graduate students and students of corresponding specialities.

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INTRODUCTION

Development of radar technology made necessary the creation of a theory of radar, which would establish the basic regularities and unique criteria of quality of radar systems.

This theory, in accordance with the functions performed by any radar set, is statistical. Actually, radar is used for detection of objects and measurement of parameters of their motion. The presence or absence of an object, as well as these parameters, are random. Besides the randomness of measured magnitudes there are other causes of the randomness of input radar data — the randomness of a reflected signal (fluctuation of a signal) and the mandatory presence of those or other interferences. Such interferences can be natural noises of receivers, reflections from nearby objects, the surface of the earth and/or sea, and also specially organized interferences of various forms employed for combat with systems of military assignment. As a result, the received radar signal is random and the radar set revealing the object or measuring its coordinate must receive those or other statistical resolutions coupled with the signal from the object (the resolution about the presence or absence of an object, or about the various values of its parameters).

The problem of analysis of the quality of radar work of every given form reduces to the investigation of random processes in it with the influence on the receiver input of a random signal mixed with noises or interferences.

The problem of synthesis of radar reduces to finding optimum (from the viewpoint of one or another applied statistical criterion) mathematical operations on the received signal and to the construction of a functional circuit executing these operations. Thus, in the synthesis of radar in the mode of detection the probability of errors of resolution are minimized (the probability of a false alarm and the omission of an object) and on this basis the method of optimum treatment of the signal is found. In the mode measuring object coordinates, randomly changing in time, frequently the mean-quadratic error of measurement at every moment of time is minimized, which is attained by the optimum filtration of the signal.

In connection with the rapid development of radar statistical analysis and the synthesis of radar systems and on their basis the development of the main regularities peculiar to radar, became the subjects of scientific investigations rather widely conducted in the past few years in various countries. First of all there appeared works devoted to the analysis of the properties of applied radar sets. One of the first and, probably, the most complete was the book "Threshold Signals" [1]. Subsequently there appeared various works in which were resolved various particular problems of the analysis of various specific forms of radar mechanisms.

First results of statistical synthesis of radar sets appeared later. The most significant of them are the works on the theory of radar detection by V. Peterson, T. Berdsal and V. Fox [2], D. Middlton [3], D. Middlton and Van Meter [4], [5], [6], V. Sibert [7]. To the development of principles peculiar to radar are devoted also the works of F. Woodward and I. Davis, the most essential results of which are presented in the book of F. Woodward [8]. Problems of the detection of signals, not only on a background of noises but also on a background of certain forms of interfering reflections, are considered in the works of L. A. Vaynshteyn and V. D. Zubakov, reflected in their book [9].

To the problems of radar measurements are devoted significantly fewer works. Besides investigations of certain particular questions coupled with specific systems for processing of signals, one should mention here the work on the synthesis of meters by S. Ye. Fal'kovich [10]. A series of interesting problems, pertaining to the theory of detection, as well as to the theory of measurements, is presented in the book of D. Middlton [77].

There should be mentioned also the directions coupled with the application of various new forms of signals for radar and with the theory of radar survey of space. The first of them is a series of articles on phase code and frequency modulation, the references to which are presented in Chapter 1. The second direction was successfully developed by Yu. B. Kobzarev and A. Ye. Basharinov [11].

The presented references on theoretical works in the field of radar are in no way full, and are done only in order to underline interest in theory of radar from the point of view of a wide range of specialists in various countries. Such interest appeared several years ago for the authors of this monograph. This was connected with the necessity of comprehension already obtained and described in informational literature as well as with the absence of many results. The theory of detection in published works was illuminated in detail enough; however, not all problems needed for practice were brought to a conclusion. The theory of radar measurements was not sufficiently developed. In any case no attempts of a single account of the main principles of radar measurements, including the resolution of a wide range of problems having a practical value in this area are known to the authors. These circumstances were the main cause of the series of investigations conducted by the authors.

This monograph is an attempt to systematically expound the main positions of statistical analysis and synthesis of radar systems. In this are used results obtained earlier, but not presented in literature in a single form, as well as, especially, the results obtained by the authors.

It is understood that the book does not pretend to be an exhausting account of all problems of radar theory. Many of them, moreover, are not even developed now. A significant part of the obtained results relating to radar are intended for work on one object. This put its own imprint on the character of the resolved problems, although many results and, all the more so, the methods of resolution of problems and certain marked general regularities relate in equal measure to all types of radar sets.

Essential to the peculiarity of the statistical approach to radar is the possibility to investigate a noiseproof feature of radar sets in reference to various applied forms of interferences inasmuch as they are random processes. Therefore the statistical analysis of radar systems by various distributive laws of probabilities of input signal, in essence, coincides with an analysis of noiseproof feature.

If, however, we are limited only to an analysis, then there always remains a known dissatisfaction connected with the problem about the fact that it is impossible to obtain better characteristics (range, accuracy, resolving power, noiseproof feature, etc.) than given by the analyzed systems.

In connection with this, special value is obtained by the synthesis of optimum systems which in given conditions are the best (best qualities of detection, highest accuracy, etc.). In a number of cases it appears that the existing forms of systems possess properties very near to the properties of optimum systems. This indicates the fruitlessness of empirical attempts to find the best forms of systems. In other cases it appears that optimum systems ensure rather large gains not realized in practice. In these cases, theoretical synthesis directly promotes the obtaining of new properties. Furthermore, analysis of absolutely all possible modifications of radars is apparently impossible. Because of these considerations, the authors took the synthesis of optimum radar systems as a basis in the book.

Statistical synthesis is presented in accordance with the functions performed by radar, since, by only specifying these functions manages to formulate a specific criteria of optimumness, and to bring the problem of synthesis to a conclusion.

In light of modern conceptions, any radar set performs two main functions: detection of targets and measurement of the parameters of their motion. The first of these functions is investigated in detail in the first volume of the book, the second--in the second volume. The problems of detection include both detection of a reflected signal on a background of noises and interferences in every point of parameter spaces of signal, and coverage of space or search in the regions of space defined by data of external target indication. The problems of measurement included the separate measurement of parameters of the motion of the object (range, velocity, angles) and their join measurement.

The authors tried to illuminate all the main problems in the following plan. After a discussion of the main initial sample there is produced a synthesis of an optimum system and its properties are investigated, then there is produced an analysis of the systems near to optimum, and also practically applied systems for the purpose of their comparison.

Synthesis of optimum mechanisms is conducted in most cases on the assumption that the reflected signal represents a normal random process and is mixed with white noise or with some other normal process appearing due to interfering reflections. Analysis and synthesis in other assumptions about the signal have a fragmentary character.

For synthesized systems an analysis is produced of the noiseproof feature in reference to certain forms of interferences. This analysis, in view of the difficulty of corresponding calculations, is not equally conducted everywhere in detail.

For convenience of discussing the problems of analysis and synthesis of various forms of radar systems they are first considered for a case of the reception of a coherent signal, and then an incoherent one.

In the book there is taken the following order of discussion.

Chapter 1 is devoted to radar signals and interferences. In it are presented the bases needed for the construction of the theory, characteristics of the various forms of main signals, the properties and signals received are described.

In the analysis of various forms of radar sets it is always necessary to use the results of influence of signals and interferences on the main elements of the receiving mechanisms (amplifiers, detectors, receivers with automatic gain control, etc.). In order to avoid numerous repetitions, problems of influence of signals and interferences on these elements are considered in Chapter 2.

In Chapter 3 there are expounded general problems of the theory of radar detection. In it, first of all, are presented several bases of the theory of statistical resolutions which is the theoretical base of the synthesis of radar systems. On the basis the general aspect of theory of detection of objects in space is considered and it appears that in all cases it is sufficient to produce detection "along the points" of this space. There are revealed several general regularities of such detection and problems of the optimization of space coverage and investigation of the object are considered.

Chapter 4 is devoted to the detection of a coherent signal, which is investigated on the basis of the general results of Chapter 3. First of all considered is the detection of signals on a background of noises: optimum systems of detection are synthesized and various methods are investigated of their realization. Investigated are dependences of characteristics of detection on the width of the spectrum of fluctuations of the reflected signal, the number of utilized carrier frequencies, and various deviations from optimum treatment of the signal inevitable in the practical realization of a system. Analogous problems are also investigated for cases where, besides noise, there are passive and active interferences. Furthermore, there are considered problems on the possibility of improving the characteristics of detection by means of rational selection of a law of signal modulation.

Analysis and synthesis of systems of detection of incoherent signal constitute the contents of Chapter 5. Synthesis of optimum systems leads, in a given case, to a conclusion with more particular conditions than with the detection of a coherent signal. Here systems are considered with an accumulation of squares of values of received signal envelope, with a binary accumulation and with the integration of range scanning.

The enumerated chapters constitute the contents of the first volume of the book. The second volume starts with Chapter 6, devoted to the general principles of radar measurements. In the discussion of the problems of coordinate measurement, in light of statistical theory, a special stop is built on the tracking radar meters, finding wide distribution. Statistical analysis of nontracking meters is conducted in less detail. The main attention in the chapter is allotted to the synthesis of optimum radar meters. Besides the discussion of the possible aspects of such a synthesis there is consecutively brought to a conclusion the synthesis of an arbitrary meter with the Gaussian statistics of measured parameters. With respect to optimum discriminators, only general conclusions are drawn; specific resolutions are considered in subsequent chapters. Optimum smoothing circuits are synthesized in the chapter in a general form.

In Chapters 7 and 8, on the basis of general theory presented in Chapter 6, problems are investigated of range measurement with a coherent and an incoherent signal accordingly. The main attention is allotted to synthesis of optimum discriminators and consideration of various technical variants of discriminators. Consideration is given to an arbitrary modulated signal, and then the results are made specific for all possible applied forms of modulation. Problems are investigated of the accuracy of range finders with various discriminators and various smoothing circuits, synthesized in Chapter 6. There is considered the influence of interferences on range finders, switching the nonlinear phenomena in them. An analysis is also made of nontracking range meters.

In Chapter 9 there are conducted investigations of velocity meters based both on the use of the Doppler effect, and on the range differentiation and the angular coordinates. First of all the optimum frequency discriminators are synthesized, and the various possible systems are considered for their technical realization with an analysis of the characteristics of these systems. Problems are investigated of the accuracy of meters velocity, on the whole, with various forms of discriminators and smoothing circuits, and also with the application of various principles of measurement. There are considered certain problems connected with the action of interferences on velocity meters.

In Chapter 10 there is considered the measurement of angular coordinates with a coherent signal, and in Chapter 11 -- with an incoherent signal. The distinctive peculiarity of these measurements as compared with measurements of range and velocity is the fact that analysis and synthesis of goniometers are conducted for various methods of angular direction finding. With this there is synthesized an optimum radio channel (discriminator), the form of which is specified for every method of direction finding and systems near to optimum are investigated with the influence of both noises and some forms of interferences. An analysis is made not only of discriminators, but also of the accuracy of goniometers on the whole. Certain nonlinear phenomena in goniometers are investigated.

In Chapter 12 optimum meters of several, in general, interdependent parameters of the motion of the object (objects), are synthesized at once. It appears that the optimum parameter totality meter is not divided, as a rule, into unconnected meters of separate parameters, but represents a certain complicated multichannel system, the channels of which are interconnected. The general solutions found are used for the resolution of some particular problems.

In Chapter 13 the resolving power of radar sets is investigated in the mode of detection and measurement of target coordinates. In it, first of all, are expounded the possible approaches to an analysis of the resolving power and synthesis

of optimum power from the viewpoint of the solution of targets of radar sets. There are also presented possible criteria of appraisal of the resolving power. With the help of the developed methods the resolving power of several forms of systems is analyzed and optimum systems are synthesized.

It is necessary to emphasize that although the book embraces a significant quantity of problems of statistical theory of radar, placed as much as possible in a general form, by no means were all problems considered; certain discussions of actual but still unsolved problems are contained in the conclusions to Chapters.

CHAPTER 1

SIGNALS AND INTERFERENCE IN RADAR

1.1. Introductory Remarks

For a satisfactory resolution of the problem of analysis and synthesis of radar systems it is necessary to consider the real properties of radar signals. In the first place this concerns the random character of the received signal, stipulated both by fluctuations of the signal reflected from the target (appearing because of vibrations, soundings and other irregular components of the motion of the target), and fadings of the signal because of measurements of the propagational conditions of radio waves. In connection with this, in this chapter the statistical characteristics are considered of a reflected radar signal, mainly coupled with the process of reflection of the signal from the target, which is a characteristic peculiarity of radar. Furthermore there are briefly described the properties of a sounding radar signal determining in many cases the quality of work of various radar systems. In the chapter is also discussed a mathematical description of the received radar signal which, to a sufficient degree, reflects its real properties determined by experimental means.

Under the conditions of the practical use of radar systems, one of the main characteristics is their noiseproof feature in reference to natural interferences (set noises, reflections from earth and water surfaces, etc.) specially organized interferences. Therefore, here is also presented a brief description of various forms of interferences.

1.2. The Main Radar Signal

Modern radar stations are presented high requirements, which reduce to long range reception with specified characteristics and target detection time, high resolving power with a maximum number of target parameters and accurate tracking by various coordinates. The necessity for fulfillment of these sometimes contradictory requirements puts definite limitations on the character of the main signal; in a number of cases for different operating modes of radar, different forms of signal modulation are used.

Below will be briefly described some forms of main radar signals utilized at present the merits and deficiencies of separate forms of modulation shown and their influence noted on the characteristics of radar systems. Many of these problems will be considered more specifically in subsequent chapters.

1.2.1. Modulation and Coherence of Main Signal

In virtue of the number of causes, connected mainly with the conditions of propagation and with the construction of antenna systems, in radar super-high frequency oscillations are used as main signals. Such oscillations allow, by Doppler shift to determine the radial speed of the radar target; however, for determining other characteristics of its motion (range and angular coordinates) it is necessary to introduce a modulation of these oscillations. In a general case high-frequency oscillations can be subjected to amplitude and frequency (or phase) modulation. In accordance with this, for a radar signal, we have the following notation:

$$\begin{aligned} x(t) &= \operatorname{Re}\sqrt{2P_0}\,u_a(t)\exp\{i[\omega_0 t+\varphi_0+\psi(t)]\} = \\ &= \operatorname{Re}\sqrt{2P_0}\,u(t)\exp\{i[\omega_0 t+\varphi_0]\}. \end{aligned} \qquad (1.2.1.)$$

Now and henceforth the law of amplitude modulation is designated by $u_a(t)$, the function $\psi(t)$ represents the law of phase modulation, and its time derivative $\omega(t)=\frac{d}{dt}\psi(t)$ can be considered as the law of frequency modulation. It is also convenient by $u(t)=u_a(t)\exp\{i\psi(t)\}$ to designate the complex law of modulation.

By ω_0 and φ_0 we will designate the carrier frequency and the initial phase of the radar signal, and by P_0 -- its mean strength.

As a main radar signal we will mean a signal already emitted into space. With such a definition, as modulation of the main signal it is necessary to consider modulation created by the transmitting mechanism, as well as modulation superimposed on the transmitted signal by the antenna system. The latter appears, for example, during scanning (movement) of the transmitting antenna of the radar, set relatively directed on target.

Modulation of the main signal is accomplished by means of changing one or more parameters of the high-frequency carrier oscillation. Usually such a change is periodic and is characterized by its own period T_r. An important form of modulation of the main signal is amplitude pulse modulation. In a number of cases this modulation is combined with an additional frequency or phase and refer at the same time to intrapulse modulation. A description of separate forms of modulation and their main characteristics will be presented below.

Along with the conception of modulation below the conception of coherence of the main signal is considered. We will call the signal coherent, in which random changes (jumps) of the phase of the high-frequency filler are absent. This definition, obviously, also embraces signals with known phase jumps; when these jumps are eliminated during reception thanks to so-called coherent heterodyning, with which the main signal is used, for example, in the appropriate way as heterodyne voltage shifted in frequency and time.

In the frame work of the given definition continuous emission, during which it is possible to disregard various instabilities of the operating modes of the transmitter, is always coherent. As applied to a pulse signal, coherence corresponds to a simple connection between the values of the initial phase of the following one-after-the-other pulses.

A coherent pulse signal is usually formed by means of gating the amplifying chain of a transmitter, to which is fed a high-frequency oscillation from the master oscillator preliminarily filtered through corresponding frequency multipliers. Through this the pulses appear as if cut from one continuous sinusoid.

In another method of forming a pulse signal the transmitter generator (for example a magnetron) is started by the video pulses of a synchronizer, and the value of the initial phase of the high-frequency filling of adjacent pulses (owing to set noise and also various instabilities in the transmitter) appear to be random. In the absence of the above noted coherent heterodyning, such a pulse signal will be called incoherent.

The randomness of high-frequency phase of adjacent pulses of an incoherent signal does not allow to separate those phase changes of signal reflected from the target which are connected with its motion. As a result the measurement of Doppler shift becomes impossible, i.e., direct measurement of the radial velocity of the target, which is one of the essential deficiencies of an incoherent signal.*

1.2.2. Function of the Autocorrelation of the Main Signal

One of the most important characteristics of a main signal is its function of autocorrelation serving as a measure of that orthogonality of the initial signal and displaced in the time and frequency of signals, which ensures application of the given form of modulation. The value, added to this characteristic of the main signal, is explained by the fact that, as will be shown in subsequent chapters, the formation of this function is reduced in essence to the radiotechnical operation in different radar receivers which produce the multiplication of the received

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*In a number of cases it appears to be possible to measure the Doppler shift during an incoherent signal (during the time of one pulse), but the accuracy of such measurement usually is small. This is stated in more detail in Chapter 9.

signal by the required and subsequent integration for a decrease of the influence of noises.*

Let us consider some general properties of the function of autocorrelation.

In accordance with that presented, the function of autocorrelation of a main signal will be called

$$C(\tau, \Omega)=\frac{1}{T_{эф}}\int_{-\infty}^{\infty} u(t+\tau)u^{*}(t)e^{i\Omega t}dt=$$

$$=\frac{1}{T_{эф}}\int_{-\infty}^{\infty} u_{a}(t+\tau)u_{a}(t)e^{i\Omega t+i\psi(t+\tau)-i\psi(t)}dt, \qquad (1.2.2)$$

whereby C(0, 0) = 1, and the effective duration of the signal

$$T_{эф}=\int_{-\infty}^{\infty}|u(t)|^{2}dt. \qquad (1.2.3)$$

In a number of cases, particularly in problems of measuring coordinates, it is convenient to consider the duration of the signal as unlimited. By the autocorrelation function we will understand

$$C(\tau, \Omega)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} u(t+\tau)u^{*}(t)e^{i\Omega t}dt. \qquad (1.2.2')$$

The function C(τ, Ω) is easy to express also by the spectrum of modulation of the main signal. Indeed, substituting into (1.2.2), instead of u(t) the inverse Fourier transform from the spectrum of modulation

$$U(\omega)=\int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt, \qquad (1.2.4)$$

we have

$$C(\tau, \Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{U(\omega)U^{*}(\omega+\Omega)}{T_{эф}}e^{i\omega\tau}d\omega. \qquad (1.2.5)$$

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* Time τ and frequency Ω shifts of the received signal are connected with the propagation time of the signal to the target and back and with the Doppler effect during reflection of the signal from the moving target.

Thus, the Fourier transform from the function of autocorrelation $C(\tau, \Omega)$ is written in the form

$$S_{\text{м}}(\omega, \Omega) = \frac{U(\omega)U^*(\omega+\Omega)}{T_{\text{эф}}}. \qquad (1.2.6)$$

For development of the general properties of the function $C(\tau, \Omega)$ it is convenient to introduce a conception of its effective width along both axes [$\tau_{\text{эф}}(\Omega)$ and $\Omega_{\text{эф}}(\tau)$ accordingly], which reflects the general character of the drop of function of the autocorrelation in these directions. The effective width $C(\tau, \Omega)$ along the axis τ is determined by the formula

$$\tau_{\text{эф}}(\Omega) = \int_{-\infty}^{\infty} |C(\tau, \Omega)|^2 d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} |S_{\text{м}}(\omega, \Omega)|^2 d\omega. \qquad (1.2.7)$$

Turning to the expression for the function $S_{\text{м}}(\omega, \Omega)$ we see that

$$|S_{\text{м}}(\omega, \Omega)|^2 = \frac{|U(\omega)|^2 |U(\omega+\Omega)|^2}{T^2_{\text{эф}}} = S_{\text{м}1}(\omega) S_{\text{м}1}(\omega+\Omega), \qquad (1.2.8)$$

where $S_{\text{м}1}(\omega) = S_{\text{м}}(\omega, 0)$ can be considered as spectrum of the function $C_1(\tau) = C(\tau, 0)$.

Then formula (1.2.7) can be rewritten in the form

$$\tau_{\text{эф}}(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\text{м}1}(\omega) S_{\text{м}1}(\omega+\Omega)\, d\omega. \qquad (1.2.7')$$

Hence, in particular, it follows that in the case where the function of autocorrelation $C(\tau, \Omega)$ has, along the axis τ, the form of a short pulse (so that the spectrum of $S_{\text{м}1}(\omega)$ appears to be wide and during real Doppler shifts $S_{\text{м}1}(\omega+\Omega) \approx S_{\text{м}1}(\omega)$, the effective duration of the function of autocorrelation $\tau_{\text{эф}}(\Omega)$ will depend weakly on the shift along the axis Ω.

For a determination of the effective width of the function of autocorrelation $C(\tau, \Omega)$ along the axis Ω it is useful to note that, as follows from (1.2.2), this function can be considered as a Fourier transform relative to Ω function of $\frac{1}{T_{\text{эф}}} u(t+\tau)u^*(t)$. Taking this into account we will obtain

$$\Omega_{\text{эф}}(\tau) = \int_{-\infty}^{\infty} |C(\tau, \Omega)|^2 d\Omega = \frac{2\pi}{T^2_{\text{эф}}}\int_{-\infty}^{\infty} |u(t+\tau)u^*(t)|^2 dt. \qquad (1.2.9)$$

Hence, in particular, it follows that the effective width of the function of autocorrelation along the axis Ω depends only on the amplitude modulation of the main signal.

It is obvious that in calculating the influence of noises and fluctuations of the radar signal the signals, characterized by the parameters (τ, Ω) and (0, 0) are practically indiscernible if the value of the function of autocorrelation $C(\tau, \Omega)$ differs little from $C(0, 0)$. If the function $C(\tau, \Omega)$ slowly drops from the origin of coordinates then by the same causes it will be difficult to indicate which are the true target position data within the limits of the region (τ, Ω) for which $C(\tau, \Omega)$ is approximately constant. In connection with this the square of the modulus of the function of autocorrelation $|C(\tau, \Omega)|^2$ will be called the function of indeterminacy of the main signal. In form of function of indeterminacy it can be judged against the resolving power and accuracy which can ensure the application of one or another form of modulation.

The main property of the function of indeterminacy is the constancy limited by its volume. Actually,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|C(\tau, \Omega)|^2 d\tau d\Omega = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S_{u1}(\omega)$$

$$\times S_{u1}(\omega + \Omega)\, d\Omega\, d\omega = \frac{1}{(2\pi)^2}\left[\int_{-\infty}^{\infty}S_{u1}(\omega)\, d\omega\right]^2 =$$

$$= \frac{1}{(2\pi T_{э})^2}\left[\int_{-\infty}^{\infty}|U(\omega)|^2 d\omega\right]^2 = |C(0, 0)|^2 = 1. \qquad (1.2.10)$$

This property of the function of indeterminacy received the name of "the indeterminacy principle in radar" [7]. In accordance with this principle it is impossible to arbitrarily decrease the "volume of indeterminacy" limited by the function of indeterminacy it can only be deformed (compressed along one of the axes owing to the extension along the other) or redistributed on the plane (τ, Ω)

In the latter case there appear additional maxima of functions of indeterminacy which lead to an ambiguity in the determination of the target coordinate data. Such a redistribution is most vividly developed with a periodic signal; connected with it, the ambiguity is physically explained by the fact that the delay of signal on the period of modulation does not lead to a change of any characteristics of signal, and a frequency shift to a frequency of repetition of modulation when it, as frequently occurs, is considerably smaller than the width of the spectrum of modulation; it also does not practically lead to a change of signal.

Let us consider what form the function of autocorrelation of the main signal has during periodic modulation. In real conditions the total duration of the periodic signal is limited (for example, owing to the movement of the transmitting antenna). Introducing, in connection with this, the conception about the envelope of the periodically modulated signal $g(t)$, so that $u(t) = g(t)u_0(t)$ (where $u_0(t)$ is periodic signal modulation with period T_r), an expression for the function of autocorrelation of such a signal can be written in the form

$$C(\tau, \Omega) = \frac{1}{T_{\text{эф}}} \int_{-\infty}^{\infty} g(t+\tau)\, g(t)\, u_0(t+\tau)\, u_0^*(t)\, e^{i\Omega t} dt =$$

$$= \frac{1}{T_{\text{эф}}} \sum_{k=-\infty}^{\infty} \int_0^{T_r} g(kT_r + t + \tau)\, g(kT_r + t)\, u_0(t+\tau)\, u_0^*(t) \times$$

$$\times e^{i\Omega(kT_r + t)} dt = \frac{1}{T_{\text{эф}}} \sum_{k=-\infty}^{\infty} g(kT_r + \tau)\, g(kT_r) \times$$

$$\times e^{i\Omega kT_r} \int_0^{T_r} u_0(t+\tau)\, u_0^*(t)\, e^{i\Omega t} dt.$$

Here we divided the interval of integration into periods of modulation, considered the periodicity of modulation, because of which $u_0(kT_r + t) = u_0(t)$, and also the smallness of change of the envelope $g(t)$ after the period of modulation T_r.

Let us designate by $C_0(\tau, \Omega)$ the function of autocorrelation of one period of modulation

$$C_0(\tau, \Omega) = \frac{1}{T_r} \int_0^{T_r} u_0(t+\tau)\, u_0^*(t)\, e^{i\Omega t} dt. \qquad (1.2.11)$$

Let us note henceforth that the function $C_0(\tau, \Omega)$ is periodic in τ with period T_r.

Introducing the spectrum of the envelope of the periodically modulated signal

$$G(\omega)=\int_{-\infty}^{\infty} g(t)\,e^{-i\omega t}\,dt,$$

the function of autocorrelation of this signal can be rewritten in the form

$$\begin{aligned}
C(\tau, \Omega) &= \frac{C_0(\tau, \Omega)}{2\pi T_{\text{эф}}}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 G(\omega_1)\,G^*(\omega_2)\,\times \\
&\times e^{i\omega_2\tau}\,\frac{T_r}{2\pi}\sum_{k=-\infty}^{\infty} e^{i(\omega_1-\omega_2+\Omega)kT_r} = \\
&= \frac{C_0(\tau, \Omega)}{2\pi T_{\text{эф}}}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 G(\omega_1)\,G^*(\omega_2)\,e^{i\omega_2\tau}\sum_{k=-\infty}^{\infty}\delta\Big(\omega_1 - \\
&- \omega_2+\Omega-k\frac{2\pi}{T_r}\Big) = \frac{C_0(\tau, \Omega)}{2\pi T_{\text{эф}}}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} G\Big(\omega-\Omega + \\
&+ k\frac{2\pi}{T_r}\Big)\,G^*(\omega)\,e^{i\omega t}\,d\omega,
\end{aligned} \qquad (1.2.12)$$

where $\delta(x)$ is the delta-function.

For manifestation of form of function of autocorrelation it is convenient to assume that the envelope $g(t)$ of the considered periodic signal is right-angled. By an increase in the duration of the signal $T_{\text{эф}}$ there occurs a narrowing of the central peak of function $G(\omega)=\dfrac{\sin\omega\frac{T_{\text{эф}}}{2}}{\frac{\omega}{2}}$ which is the spectrum of $g(t)$. Using the "filtering" property of this function we obtain for the function of autocorrelation of the periodic signal the following expression:

$$C(\tau, \Omega)\approx C_0(\tau, \Omega)\sum_{k=-\infty}^{\infty}\frac{\sin\frac{(\Omega-k\omega_r)T_{\text{эф}}}{2}}{\frac{(\Omega-k\omega_r)T_{\text{эф}}}{2}}.$$

For signals with high resolving power with respect to range for which $C_0(\tau, \Omega)$ weakly drops along the axis Ω, the function of autocorrelation $C(\tau, \Omega)$ during periodic modulation represents as a function of Ω the totality of values of $C_0(\tau, \Omega)$,

taken by intervals equal to the frequency of repetition of modulation $\omega_r = \frac{2\pi}{T_r}$. Because of limited duration of the periodically modulated signal the peaks of function of autocorrelation $C(\tau, \Omega)$ have along the axis Ω a finite width on the order of $\frac{1}{T_{\text{эф}}}$. With a fixed Ω $C(\tau, \Omega)$ is in accordance with (1.2.11) periodic function τ with a period T_r. The general form of the function $C(\tau, \Omega)$ for periodic signal is illustrated in Fig. 1.1.

In conclusion, let us make one remark concerning the function introduced by us for the autocorrelation of one period of modulation $C_0(\tau, \Omega)$. as it is easy to see,

$$C_0(\tau, \Omega) = \frac{1}{2\pi T_r} \int_{-\infty}^{\infty} U_0^*(\omega) e^{i(\omega - \Omega)\tau} d\omega \int_{\tau}^{T_r + \tau} u_0(x) e^{-i(\omega - \Omega)x} dx \approx$$

$$\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{U_0(\omega) U_0^*(\omega + \Omega)}{T_r} e^{i\omega\tau} d\omega, \qquad (1.2.13)$$

where $U_0(\omega)$ is the spectrum of one period of modulation.

$$U_0(\omega) = \int_0^{T_r} u_0(t) e^{-i\omega t} dt. \qquad (1.2.14)$$

Fig. 1.1. Function of indeterminacy $|C(\tau, \Omega)|^2$ for a periodic signal. $\tau_{\text{эф}}$ is the width of the function $|C_0(\tau, \Omega)|^2$.

KEY: (a) эф effective.

The approximate equality in (1.2.13) is correct under the condition that $C_0(\tau, \Omega)$ sufficiently rapidly diminishes with the growth of $|\tau|$.

A comparison of (1.2.5) and (1.2.13) shows that the function $C_0(\tau, \Omega)$ in the interval equal to the period of modulation possesses the same general properties as the above considered function of the autocorrelation of the main signal with nonperiodic modulation.

1.2.3. Frequency Modulation

Proceeding to a consideration of the separate forms of modulation, we will describe their characteristics with the help of the above introduced function of autocorrelation, reflecting all the properties of the main signal of interest to us. In this and following points will be considered the various forms of modulation of a continuous signal.

As follows from the preceding [see (1.2.7)] the effective width of the function of autocorrelation $\tau_{\text{эф}}(0)$ along the axis τ , and accordingly the resolving power with respect to range, depend on the width of the spectrum of modulation. One of the possible methods of expanding the spectrum of a continuous main signal is the introduction of modulation of its frequency. Frequency modulation was historically the first method for obtaining resolving power with respect to range during continuous emission.

With frequency modulation, the frequency of the main signal can be represented in the form

$$\omega(t)=\omega_0+\omega_{\text{м}}(t), \tag{1.2.15}$$

where

$$\omega_0=\frac{1}{T_r}\int\limits_0^{T_r}\omega(t)\,dt; \tag{1.2.16}$$

T_r is the period of modulation.

In case of sinusoidal frequency modulation the frequency of the signal changes according to the law

$$\omega_{\text{м}}(t)=\omega_{\text{м}}\sin\omega_r t.$$

and the phase of the signal--according to the law

$$\psi(t)=\int\limits_0^t \omega_{\text{м}}(t)\,dt=\frac{\omega_{\text{м}}}{\omega_r}(1-\cos\omega_r t).$$

Autocorrelation function for case as is easy to see, has the form

$$C_{\text{ф}}(\tau,\Omega)=\frac{1}{2\pi}\int\limits_0^{2\pi} e^{ia\sin\left(\varphi+\frac{\omega_r\tau}{2}\right)+ib\varphi}\,d\varphi, \qquad (1.2.17)$$

where $\omega_r=\frac{2\pi}{T_r}$ is the angular frequency of the repeated modulation;

$$b=\frac{\Omega}{\omega_r};\quad a=2\frac{\omega_{\text{м}}}{\omega_r}\sin\frac{\omega_r\tau}{2}.$$

To take the integral (1.2.17) in the general form, unfortunately, is impossible; therefore, we are forced to limit ourselves to the consideration of $C_{\text{ф}}(\tau,\Omega)$ for certain characteristic cases. On the axis Ω

$$C_{\text{ф}}(0,\Omega)=\frac{e^{i2\pi b}-1}{i2\pi b}=\frac{e^{i\Omega T_r}-1}{i\Omega T_r},$$

i.e., $|C_{\text{ф}}(0,\Omega)|^2$ drops along the axis Ω, as $\frac{2}{\Omega T_r}(1-\cos\Omega T_r)$. For $b=\frac{\Omega}{\omega_r}=k$ (k is an integer) the expression (1.2.17) coincides with accuracy up to the coefficient with an integral presentation of the Bessel function [12], so that

$$C_{\text{ф}}(\tau,k\omega_r)=(-1)^k J_k(a)\,e^{-ik\frac{\omega_r\tau}{2}},$$

where $J_k(a)$ is a Bessel function of the k-th order of the 1-st sort.

Usually $\frac{\omega_{\text{м}}}{\omega_r}\gg 1$. In this case

$$|C_{\text{ф}}(\tau,k\omega_r)|^2\simeq J_k^2(\omega_{\text{м}}\tau).$$

the contour of the function of indeterminacy for sinusoidal FM is represented in Fig. 1.2 in the form of curves $J_k^2(\omega_{\text{м}}\tau)$ = const on the planes $(k,\omega_{\text{м}}\tau)$. As can be seen from the figure the resolving power assured by the use of a harmonic FM is rather low: at $\Omega=0$ (on axis τ) are secondary maxima for $\tau\neq0$; the first magnitude of these maxima constitutes approximately 16% of the base. With an increase in Ω there takes place a displacement of the main maximum along the curves

with real values of ω_M and T_r this slope is quite small, so that the presence of the region of indeterminacy shows up mainly in the resolving powers with respect to frequency. The width of the interval of frequencies, within the limits of which the solution is impossible, approximately coincides with the width of the spectrum of modulation and for the case considered is approximately equal to ω_M.

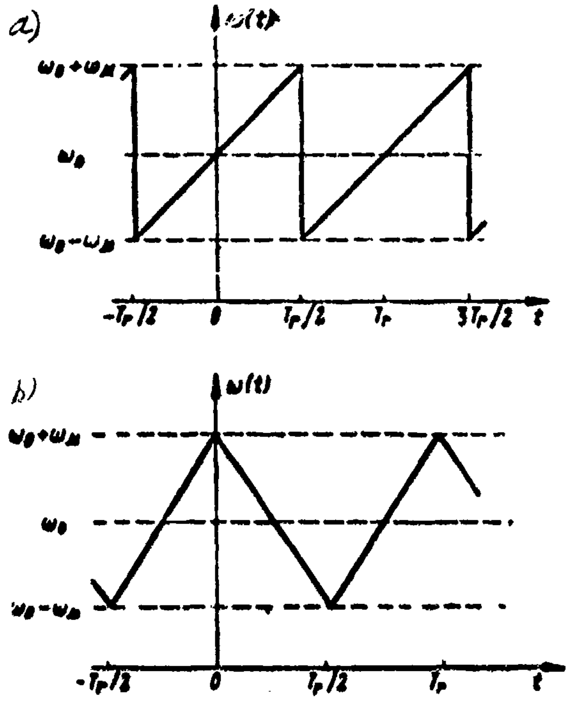

Fig. 1.3. Change of frequency of continuous main signal with linear frequency modulation: a) asymmetric sawtooth; b) symmetric sawtooth.

The width of the region of ambiguity along the axis τ, is naturally equal to $\frac{2\pi}{\omega_M}$. Graphically the function $|C_0(\tau, \Omega)|^2$ with linear frequency modulation is illustrated by Fig. 1.4.

In the case of frequency modulation, according to the law of symmetric sawtooth waves

$$\omega(t) = \omega_0 + \omega_M\left(1 - \frac{4|t|}{T_r}\right) \quad \text{at } |t| < \frac{T_r}{2}. \quad (1.2.20)$$

Using the same assumption on the smallness of τ, we obtain

$$C_0(\tau, \Omega) \approx \frac{\exp\left\{i\frac{\Omega T_r}{2}\right\}}{2}\left[e^{i\frac{\Omega T_r}{4}\tau}\frac{\sin\left(\omega_M\tau - \frac{\Omega T_r}{4}\right)}{\omega_M\tau - \frac{\Omega T_r}{4}} - e^{-i\frac{\Omega T_r}{4}\tau}\frac{\sin\left(\omega_M\tau + \frac{\Omega T_r}{4}\right)}{\omega_M\tau + \frac{\Omega T_r}{4}}\right]. \quad (1.2.21)$$

There are two main zones of indeterminacy located along lines

$$\Omega = \pm\frac{4\omega_M}{T_r}\tau,$$

the magnitude of the slope of which, as before, is equal to the speed of change of frequency during modulation. The magnitude of the function of indeterminacy $|C_0(\tau, \Omega)|^2$ in the zone of indeterminacy (on each of these lines) diminishes with $\sqrt{2}\left|\frac{\Omega T_r}{4}\right| > \pi$ to 0.25 (in case of asymmetric sawtooth $|C_0(\tau, \Omega)|^2$ kept the unit value in this region).

Thus, the above are obtained formulas for the function of indeterminacy $|C_0(\tau, \Omega)|^2$ corresponding to one period of modulation with different forms of

near to $\frac{\Omega}{\omega_r} = \pm \omega_{\text{м}} \tau$; secondary maxima are displaced almost parallel to the base. Extent of these maxima along the axis τ can be estimated by a magnitude of the order $\frac{2\pi}{\omega_{\text{м}}}$.

From other possible functions of $\omega_{\text{м}}(t)$ the most frequently used is the linear change of frequency by the laws of asymmetric and symmetric sawtooth waves (Fig. 1.3)

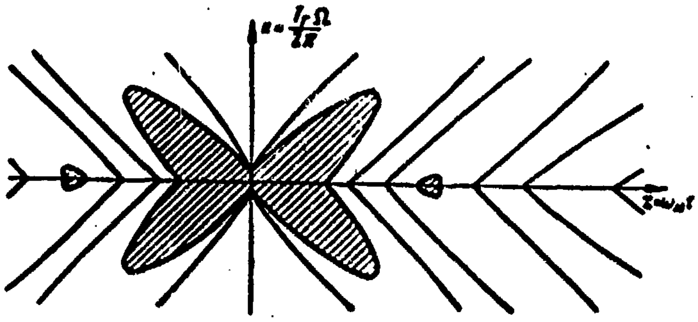

Fig. 1.2. Section of the function of indeterminacy $|C_0(\tau, k\omega_r)|^2$ for sinusoidal frequency modulation of a continuous main signal.

In the first case, with a continuous main signal we have:

$$\left.\begin{aligned} \omega(t) &= \omega_0 + \omega_{\text{м}} \frac{2t}{T_r} \quad \text{at } |t| < \frac{T_r}{2}, \\ \psi(t) &= \omega_{\text{м}} \frac{t^2}{T_r}. \end{aligned}\right\} \tag{1.2.18}$$

The function of autocorrelation is determined by formula (1.2.12). Replacing for a small τ the difference $\psi(t+\tau) - \psi(t)$ by $\frac{d\psi(t)}{dt}\tau = \omega_{\text{м}}(t)\tau$, we obtain

$$C_0(\tau, \Omega) = \exp\left\{i \frac{\Omega T_r}{2}\right\} \frac{\sin\left(\omega_{\text{м}}\tau + \frac{\Omega T_r}{2}\right)}{\omega_{\text{м}}\tau + \frac{\Omega T_r}{2}}. \tag{1.2.19}$$

As can be seen from the formula, $|C_0(\tau, \Omega)|^2 = 1$ is along straight line

$$\Omega = -\frac{2\omega_{\text{м}}}{T_r}\tau.$$

Near this line there naturally exists a region of indeterminacy , within the limits of which the solution of targets is practically impossible.* The slope of this region to the axis Ω coincides with the speed of change of frequency at FM ;

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*This region, of course, does not stretch to infinity; at large τ the utilized approximation becomes insufficient; the function of autocorrelation diminishes faster than this follows from (1.2.19).

frequency modulation. The function of indeterminacy $|C(\tau, \Omega)|^2$ for a periodic frequency modulated main signal will be formed in accordance with (1.2.12) by means of a periodic one (with the period T_r) -- repetition of the function $|C_0(\tau, \Omega)|^2$ along the axis τ, by which along the axis Ω from this function are "selected the values of $|C_0(\tau, k\omega_r)|^2$, where k is an integer. The width of the maxima formed in this was along the axis Ω is the reciprocal of total duration of the f.m. signal. These maxima lead to ambiguity in determining the parameters of the target.

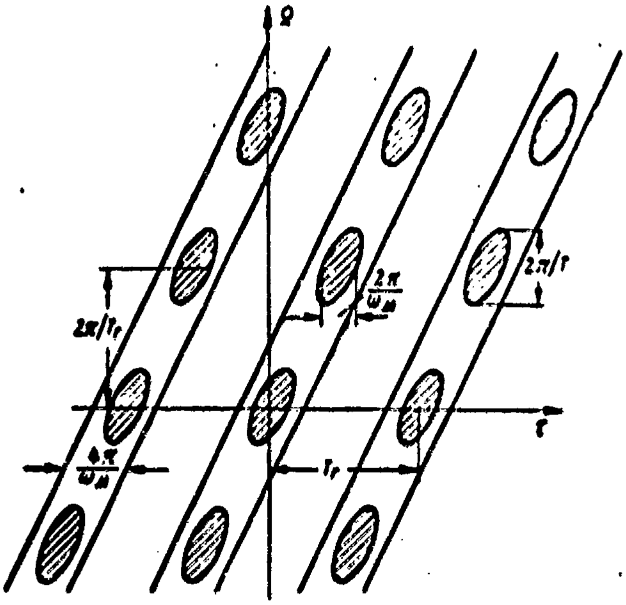

Fig. 1.4. Section of the function of indeterminacy for linear frequency modulation.

1.2.4. Phase Code Manipulation

By phase code manipulation (PCM) is understood such a modulation of the main signal with which is established a definite sequence (code) of changes of initial phase of a high-frequency oscillation in so-called code intervals, into which the whole period of modulation is divided, during continuous emission. Duration of the code intervals is constant and is equal to τ_u.*

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*It is possible, certainly, to call an interval code, within the limits of which a shown phase remains constant. The duration of every such interval, naturally depends on the code utilized. It is possible, however, to find the interval which is the greatest common divisor for all intervals. Its duration is also designated by τ_u.

Any PCM signal can be considered as the sum of n shifted one relative to another at a magnitude $\tau_и$ of elementary sequences of right-angle pulses, each of which is formed by oscillations in code intervals, equally located in adjacent periods of modulation. Within the limits of one period the modulating signal can be recorded in the form

$$\sum_{l=1}^{n} \exp\{i\psi_l\} f[t-(l-1)\tau_и],$$

where $n=\frac{T_r}{\tau_и}$;

$\psi_l=\psi(l\tau_и)$ is the relative shift of the initial phase in the l-st code interval;

$f[t]$ is the unit right-angle pulse of duration $\tau_и$.

Substituting this expression into (1.2.11), we obtain for the function of autocorrelation of one period of modulation along the axis τ the following formula:

$$C_0(\tau)=\frac{1}{n}\sum_{l=1}^{n}\Big[\Big(1-\frac{\Delta\tau}{\tau_и}\Big)\exp\{i\psi_{l+\nu}-i\psi_l\}+ \\ +\frac{\Delta\tau}{\tau_и}\exp\{i\psi_{l+\nu+1}-i\psi_l\}\Big], \qquad (1.2.22)$$

where $\tau=\nu\tau_и+\Delta\tau$.

If the shift of PCM of the signal does not exceed the duration of code interval $(\tau=\Delta\tau)$, then

$$C_0(\tau)=1-\frac{\tau}{\tau_и}\Big[1-\frac{1}{n}\sum_{l=1}^{n}\exp\{i\psi_{l+1}-i\psi_l\}\Big].$$

If the shift of PCM of the signal is a multiple of duration of code interval $(\tau=\nu\tau_и)$, then

$$C_0(\tau)=\frac{1}{n}\sum_{l=1}^{n}\exp\{i\psi_{l+\nu}-i\psi_l\}.$$

The problem of constructing a code reduces to the selection of such a sequence and such values of phases by which $C_0(\nu\tau_и)$ for all ν would be sufficiently small (in principle it is desirable to obtain $C_0(\nu\tau_и)=0$). Thus, the function of autocorrelation of the PCM of a signal has on axis τ one maximum at $\tau=0$ and, possibly, several additional maxima ("remainders") at $\tau>\tau_и$. The magnitude of these "remainders" depends considerably on the selection of the code.

The main characteristics of codes are the duration of code interval τ_{κ} determining the resolving power with respect to range at a considered main signal, and the quantity of code intervals n, determining in many cases the magnitude of the "remainders" during encoding. The required interval of uniqueness is still one factor influencing the selection of these characteristics inasmuch as $T_r = n\tau_{\kappa}$.

In the absence of the completed theory of phase code manipulation there is, however, a significant number of works devoted to this form of modulation, and noticeable successes are attained in the matter of practical construction of various codes. For continuous emission binary codes were offered (particularly described in [7]), for which the initial phases of high-frequency oscillations in adjacent code intervals differ from 0 or π, and the quantity of code intervals n is determined by the formula $n = 2^m - 1$, where m is any integer. "Remainders"[#] for such codes have a magnitude $\frac{1}{n^2}$ (Hoffman's code), since at $\tau = \nu\tau_{\kappa}$ the quantity of code intervals with coinciding values of initial phases are always of less quantity per unit of such intervals with noncoincident values of initial phases. It follows from this that for compensation of these "remainders" it is sufficient to select as the difference of initial phases a certain magnitude $\varphi_0 \neq \pi$. It is easy to see that $\varphi_0 = \pi - \arccos\left(\frac{n - 1}{n + 1}\right)$.

Phase code manipulation of the main signal has obtained lately significant circulation. In favor of phase code manipulation in its comparison with frequency modulation remaining in a certain sense classical, is the absence of indeterminacy in the treatment of frequency shift of the echo signal in measuring the range and velocity of a target and high simplicity of the creation of the corresponding coding and decoding equipment. These simplifications are connected, in the first place, with the constant duration of code intervals and with fixed changes of phase of

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[#] There are magnitudes of additional maxima of the function of indeterminacy $|C_0(\tau, 0)|^2$.

high frequency oscillations at the limits of these intervals, i.e., with allowed discretisation of the characteristics of coders and decoders.

1.2.5. Pulse Modulation

In amplitude pulse modulation the function of $u_a(t)$ is considerably different from zero only within the limits of a certain interval $\tau_и$, of a smaller period of repetition of the modulation T_r. The duration of this interval is called the pulse duration and in a general case is determined* as

$$\tau_и = \frac{1}{u^2_{a\max}} \int_0^{T_r} u_a^2(t)\,dt.$$

The general form of the function of autocorrelation of a periodic pulse signal, like a signal with any other periodic modulation, is determined by formula (1.2.12) and does not differ in the same way from that described above (Par. 1.2.2). The pulse character of the signal appears only in the form of a function of indeterminacy of one period of modulation $|\dot{C}_0(\tau, \Omega)|^2$. Let us consider as examples the amplitude modulation of the signal by a sequence of square and Gaussian pulses.

The function of indeterminacy for one period of modulation in the case of a square pulse of the duration $\tau_и$ is determined, as is easily verified, by the formula

$$|C_0(\tau, \Omega)|^2 \approx \left(1 - \frac{\tau}{\tau_и}\right)^2 - \frac{\Omega^2 \tau_и^2}{12} \left(1 - \frac{\tau}{\tau_и}\right)^4. \qquad (1.2.23)$$

corrected when $\tau < \tau_и$. For a random Gaussian pulse $\exp\left\{-1{,}57\left(\frac{t}{\tau_и}\right)^2\right\}$ the function of indeterminacy is equal to

$$|C_0(\tau, \Omega)|^2 = \exp\left\{-1{,}57\left[\left(\frac{\tau}{\tau_и}\right)^2 + \frac{(\Omega\tau_и)^2}{3{,}52}\right]\right\}, \qquad (1.2.24)$$

where $\tau_и$ is the effective pulse duration.

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* One can determine the pulse duration also by a certain level of the function $u_a(t)$.

As can be seen from these formulas, the function of indeterminacy for one period of modulation (or for a single pulse*) has a maximum in the origin of coordinates and monotonically decreases with an increase of τ and Ω (Fig. 1.1). With a decrease in pulse duration $\tau_{\text{и}}$ the drop of $|C_0(\tau, \Omega)|^2$ along the axis τ occurs more sharply so that the maximum of the function of indeterminacy in this direction is narrowed; the changes along axis Ω carry an inverse character.

As can be seen from the foregoing, obtaining a high range resolving power requires a shortening of the pulse which in turn leads during a periodic pulse signal with a limited peak power, to a lowering of its mean power. In connection with this for the expansion of the main signal spectrum, at present, instead of shortening the pulses, there is frequently applied an additional intrapulse frequency and phase code modulation.

For a frequency modulated pulse, the frequency of which changes linearly in time we have

$$u(t) = u_a(t)\exp\left\{i\alpha\frac{t^2}{2}\right\},$$

where $\alpha = \frac{d\omega(t)}{dt}$ = const is the speed of frequency change.

For one period of frequency modulated pulse signal the function of autocorrelation will be equal to

$$C_0(\tau, \Omega) = \frac{e^{i\alpha\frac{\tau^2}{2}}}{T_r}\int_{-T_r/2}^{T_r/2} u_a(t+\tau)u_a(t)e^{i(\alpha\tau+\Omega)t}dt. \qquad (1.2.25)$$

Usually the pulse duration is long compared with the width of the peak of the function of indeterminacy along axis τ:

$$\tau_{\text{и}} \gg \frac{2\pi}{\alpha\tau_{\text{и}}},$$

where $\alpha\tau_{\text{и}}$ is the deviation of frequency within limits of the pulse. In connection

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*Besides a periodic pulse signal, there is also possible a main signal in the form of an isolated sending. Such a signal will be formed, for example, during the motion of the transmitting radar set antenna of continuous emission relative to the direction considered; the form of pulse in this case is determined by form of the pattern of directivity of the antenna system. The function of indeterminacy of such a signal coincides, on the interval of duration T_r with the function of indeterminacy of one period of the corresponding periodic pulse modulation.

with this, if one were interested in $C_0(\tau, \Omega)$ near the main maximum, then $u_a(t+\tau)\approx u_a(t)$ and

$$|C_0(\tau, \Omega)|^2 \sim |U_2(\alpha\tau+\Omega)|^2, \qquad (1.2.26)$$

where $U_2(\omega)$ is the spectrum of function of $u_a^2(t)$.

As can be seen from (1.2.26), in this case, as in frequency modulation of a continuous signal, the zone of indeterminacy is along line $\tau = -\frac{\Omega}{\alpha}$, which because of large α practically coincides with the axis Ω. The formula (1.2.26) also allows to determine the law of amplitude modulation (the form of pulse), which ensures the extraction of the desired function of indeterminacy. Thus, for example, for a decrease in $|C_0(\tau, \Omega)|^2$ by Gaussian's law it is required, as a main signal, to take the sequence of pulses also of a bell form. Thus, linear intrapulse modulation of frequency allows, by means of selection of the law of amplitude pulse modulation to obtain the desired law of change of the function of indeterminacy along the axis τ.

Turning to phase code manipulation, let us note that during pulse emission it gives, in general, poorer results than during continuous emission inasmuch as in this case, with mixing of the received and pedestal impulses, part of the code intervals is not covered which leads to an increase in the residual level function $|C_0(\tau, \Omega)|^2$. There were binary pulse codes constructed [13], for which the magnitude of this level is on the order of $1/n^2$ (Barker's code*). However, such codes exist only for n = 3, 4, 5, 7, 11, 13. For large n's, which are required for the essential increase of resolving power during long pulses, this level has a magnitude 1/n, which in many cases, is insufficient. The way out of this position is, apparently, the use of a specially selected nonoptimum (from the viewpoint of detection reliability) processing of the signal during reception, which allows, however, to lower the level of minor lobes of the function of indeterminacy.

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*Barker's codes are also applicable with continuous emission.

Another way out is the application of codes with a higher base, and, in particular, a quaternary code [15], which with a pulse signal allows to receive the zero "remainders" of the function of indeterminacy.

1.2.6. Random Modulation

As was shown above, the periodic modulation of a main signal leads to the appearance of additional maxima of the function of indeterminacy $|C(\tau, \Omega)|^2$ and accordingly to an ambiguous determination of the coordinates of targets. One of the possible forms of nonperiodic modulation is random modulation.

The essential peculiarity of random modulation is the fact that the characteristics of the main signal also appear random. In particular, the function $C(\tau, \Omega)$, determined by the formula (1.2.2), can be considered as a certain transform of the random law of modulation and changes, depending upon the realization of the main signal to random form.

$\dot{C}(\tau, \Omega)$ could have been most fully characterised with the help of a multidimensional distributive law, however by far this law is not found in all cases. Therefore, we will henceforth be limited to the use of the simplest characteristics of the magnitude interesting us $\dot{C}(\tau, \Omega)$: the mean value, dispersion, and one-dimensional distributive law. The modulating process will be considered ergodic and stationary. The distributive laws of amplitude and phase for cases of amplitude and phase modulation will be considered normal.

Let us first turn to the case of amplitude noise modulation. For that form of modulation the mean value and dispersion of the function $C(\tau, \Omega)$ will be in accordance with (1.2.2) to be determined by the relationships

$$M_C(\tau, \Omega) = \overline{C(\tau, \Omega)} = \frac{e^{i\Omega T} - 1}{i\Omega T} \rho_x(\tau), \qquad (1.2.27)$$

$$\sigma_C^2(\tau, \Omega) = \frac{1}{T} \int_{-T/2}^{T/2} [\rho_x^2(t) + \rho_x(t-\tau)\rho_x(t+\tau)] e^{i\Omega t} dt. \qquad (1.2.28)$$

where $\rho_{M}(\tau)$ is the coefficient of correlation of the modulating process.

Usually the duration of the signal T is a much longer time than the correlation of the modulating process, so that the limits of integration in practical calculations can be spread to infinity.

The modulus of the mean value of the function of autocorrelation diminishes with an increase in $|\tau|$ and $|\Omega|$ and with a corresponding selection of $\rho_{M}(\tau)$ does not have secondary maxima which can only appear due to random blips of the function $C(\tau, \Omega)$. Dispersion of the function of autocorrelation characterizes the intensity of such blips.

The value of $C(\tau, \Omega)$ can be considered independent at points, distant from each other, by $\frac{1}{\Delta f_{M}}$ along the axis τ or by $\frac{2\pi}{T}$ along the axis Ω , where Δf_{M} is the width spectrum of the modulating process, and T is the duration of the main signal. For calculating the probability of the absence of blips of the function $|C(\tau, \Omega)|^2$, exceeding a certain level A, it is sufficient to determine the probability of absence of such blips at the points where the value of this function can be considered independent for which, in turn, it is sufficient to use a one-dimensional distribution of this function. At a large T the real and imaginary part of $C(\tau, \Omega)$ can be considered distributed by normal law. Considering them uncorrelated, we obtain for $|C(\tau, \Omega)|^2$ an exponential distribution.

Then

$$P\{|C(\tau,\Omega)|^2 \geqslant A\} = \exp\left\{-\frac{A}{\sigma_C^2}\right\}, \qquad (1.2.29)$$

where σ_C^2 is taken for the considered values of τ and Ω.

Along the axis $\Omega = 0$ the function $C(\tau, \Omega)$ is real and $|C(\tau, 0)|^2$ is distributed according to chi-square the law with one degree of freedom. At large $\frac{A}{\sigma_C^2}$ we will obtain

$$P\{|C(\tau,0)|^2 \geqslant A\} \approx \sqrt{\frac{2\sigma_C}{\pi\sqrt{A}}} \exp\left\{-\frac{A}{2\sigma_C^2}\right\}. \qquad (1.2.30)$$

The probability of the absence of blips on the parts of a plane (τ, Ω) with the dimensions $\Delta\tau \times \Delta\Omega$ will equal to

$$P_A(\Delta\tau, \Delta\Omega) \approx \prod_{j=1}^{n}(1 - P_j) \approx 1 - \sum_{j=1}^{n} P_j, \qquad (1.2.31)$$

where $n = \frac{\Delta\tau\Delta\Omega\Delta f_M T}{2\pi}$ is the number of independent values of $C(\tau, \Omega)$ in the considered part of the plane (τ, Ω);

P_j is the probability of exceeding level A at point j.

Using formulas (1.2.28), (1.2.29) and (1.2.31), one can determine (assigning a specific form of the function of correlation of the modulating process), with what values of the width of the spectrum of modulation Δf_M and the duration of the sending is T ensured with the given probability of P_A, the necessary drop of the function of indeterminacy $|C(\tau, \Omega)|^2$, characterized by level A, in the region interesting us $(\Delta\tau \times \Delta\Omega)$. In such a calculation, if one were to be interested in the order of the obtained magnitudes, the difference need not be considered between (1.2.29) and (1.2.30), and also one should take advantage of the fact that dispersion of the function $C(\tau, \Omega)$ does not depend on τ outside the basic maximum and during real Doppler shifts slightly depends on Ω, so that in the considered part $\Delta\tau \times \Delta\Omega$ can be considered constant.

Analogously consideration is given to phase noise modulation $u(t) = \exp\{j\alpha\xi(t)\}$, for which

$$\overline{u(t+\tau)u^*(t)} = \exp\{-\alpha^2\sigma_M^2[1 - \rho_M(\tau)]\}, \qquad (1.2.32)$$

where α is the coefficient characterizing the depth of modulation,

σ_M^2 and $\rho_M(\tau)$ is the dispersion and coefficient of the correlation of the modulating process $(\overline{\xi(t)} = 0)$.

The mean value of $C(\tau, \Omega)$, as before, is determined by formula (1.2.27), into which, instead of $\rho_M(\tau)$, one should substitute (1.2.32):

$$\overline{C(\tau, \Omega)} = \frac{e^{i\Omega t} - 1}{i\Omega t} \exp\{-\alpha^2\sigma_M^2[1 - \rho_M(\tau)]\}. \qquad (1.2.33)$$

For dispersion of the function of autocorrelation we will obtain

$$\sigma_C^2 = \frac{\exp\{-2a^2\sigma_\text{м}^2[1-\rho_\text{м}(\tau)]\}}{T} \int_{-\infty}^{\infty} (\exp\{a^2\sigma_\text{м}^2[2\rho_\text{м}(t) - \\ - \rho_\text{м}(t+\tau) - \rho_\text{м}(t-\tau)]\} - 1) e^{i\Omega t} dt. \qquad (1.2.34)$$

The probability of the absence of blips of the function $|C(\tau, \Omega)|^2$ in the areas of $\Delta\tau \times \Delta\Omega$ for that case can also be calculated by the formula (1.2.31). The necessary value of dispersion of σ_C^2 for calculations by this formula corresponds to the case of $|\tau| \gg \frac{1}{\Delta f_\text{м}}$, for which we have

$$\sigma_C^2 = \exp\{-2a^2\sigma_\text{м}^3\} \sum_{k=1}^{\infty} \frac{(2a^2\sigma_\text{м}^2)^k}{k!T} \int_{-\infty}^{\infty} \rho_\text{м}^k(t) e^{i\Omega t} dt. \qquad (1.2.35)$$

Assigning the form of the function of correlation of the modulating process $\rho_\text{м}(\tau)$ the final formulas for all cases interesting us can be obtained.

The considered forms of random modulations, ensuring practically identical (with an equal width of the spectrum and the time of existence) characteristics of the main signal allow, in distinction from the usually utilized forms of periodic modulation, to produce an unambiguous determination of the coordinates of targets. This proximity of random modulation to the ideal, for which the function of indeterminacy has a single maximum and uniform "remainders" on the remaining part of the plane (τ, Ω) it is correct, certainly, at a sufficiently large T.

A known difficulty in the practical use of random modulation is the necessity for storage in the receiving mechanism of the developed law of modulation, and also a lowering of the work economy of transmitters because of the random change of output power by them.

1.2.7. Multifrequency Signal

In a number of cases a totality of several signals with various carrier

frequencies is used as a main signal. Such a multifrequency signal is applied, in particular, for smoothing fluctuations of a signal reflected from the target which increases the reliability of its detection (see Chapter 4). In these cases the main signal can be recorded in the form

$$x(t)=\sum_{j=1}^{m}\sqrt{2P_j}\,\mathrm{Re}\,u_j(t)\exp\{i\omega_j t+\varphi_j\}, \qquad (1.2.36)$$

where m is the quantity of the utilized carrier frequencies;

P_j is the mean power of the j-th signal;

$u_j(t)$ is the complex law of modulation of the j-th signal.

With a multifrequency signal it is already insufficient to consider the function of autocorrelation of each of its frequency components and it is necessary to also introduce the function of their mutual correlation

$$C_{jk}(\tau,\ \omega_j-\omega_k+\Omega_{jk})=\frac{1}{T_{\text{эф}}}\int_{-\infty}^{\infty}u_j(t+\tau)\,u_k^*(t)\times$$
$$\times e^{i(\omega_j-\omega_k+\Omega_{jk})t}\,dt. \qquad (1.2.37)$$

As is shown below, in the practical use of a multifrequency signal they strive to ensure independence of the signals reflected from the target on each of the constituent frequencies which allows to produce their independent processing (the multiplication of each of the signals by the required signal and integration). The necessary (but not sufficient) condition of independence of the reflected signals is the orthogonality of the corresponding constituent of the main signal. From (1.2.37) it is clear that this orthogonality cannot be ensured by the selection of laws of modulation since the Fourier transform of the function $C_{jk}(\tau,\ \omega_j-\omega_k+\Omega_{jk})$ is the product of the spectra of modulation of the j-th and k-th signals one of which will shift in frequency to a magnitude $\omega_j-\omega_k+\Omega_{jk}$ so that the required equality to zero of this function at $j \neq k$ is attained only at a nonoverlapping of the shown spectra, i.e., at a sufficiently great magnitude of $|\omega_j-\omega_k|$. Henceforth in discussing the characteristics of various radar systems

in the case of a multifrequency signal this condition will be considered carried out.

1.3. Reflected Radar Signal

During reflection the radar signal undergoes a number of changes connected with the properties of the reflecting objects. It is natural to distinguish the useful signal received as a result of reflection from the target, the detection and measurement of the coordinates of which are a problem of the given radar set, and the signals conditioned by presence of any kind of reflecting objects camouflaging the target (passive interferences). Such objects can be the surface of the land or sea, dipole reflector clouds, hydrometeors, etc. A number of the general properties of a useful signal and interfering reflections, with certain assumptions, can be considered without specifying the form of the reflecting object which justifies the unification of these questions under discussion.

Let us start with a qualitative consideration of the properties of the signal reflected from the target. With this we will endeavor to establish a connection between the statistical characteristics of the reflected signal necessary for the future and the radar characteristics of the target usually utilized in engineering practice.

First of all one should note that the power of the reflected signal depends on the range of the target and its effective reflecting surface, the frequency and delay of signal — on the velocity and range of the target relative to the radar set. Furthermore, as it is known, the amplitude and phase of the signal reflected from the target depend on its aspect.

The dependence of the power of the echo signal on the aspect of the target is usually in the form of a pattern of secondary emission, which for the majority of radar targets at the utilized working frequencies of radar sets is strongly cut with large gaps between its extremes. For targets of a simple geometric form the

pattern of secondary emission can be calculated theoretically as a result of a more or less strict resolution of corresponding electrodynamic problems. However, for real targets, to make such a calculation is almost never managed and it is necessary to define the pattern of secondary emission experimentally.*

The patterns of secondary emission are only static characteristics of the reflected signal. Under real conditions, owing to the motion of the target and change in conditions of the propagation of radio waves, the reflected signal always fluctuates in amplitude and phase. The fluctuations, connected with the changes of propagation conditions, are not specific for radar and we will not remain on them; the fluctuations connected with the motion of the target are due to the change of aspect of the target (hunting) and the vibration of its surface. During hunting the pattern of secondary emission is turned in a random manner relative to the direction to the radar set, and owing to the vibrations changes the form of this pattern, which, as a result, appears to be a random function of time.

In virtue of all the shown causes, the signal reflected from the target is a random process and can be written as

$$x(t)=\mathrm{Re}\sqrt{2P_c}E(t)u(t-\tau)\exp\{i(\omega_0+\omega_{\text{д}})t+i\varphi(t)\}, \qquad (1.3.1)$$

where P_c is the power of signal;

$u(t)$ is the complex law of modulation introduced in the preceding paragraph;

τ and ω_A are the delay and Doppler shift of the reflected signal frequency;

$E(t), \varphi(t)$ are the laws of amplitude and phase fluctuations of the echo signal.

Formula (1.3.1) is approximate and is true with an assumption of sufficiently small dimensions of target, when "erosion" of the law of modulation with a reflection from its separate parts can be disregarded.

An adequate description of the random process $x(t)$ is possible only with the inclusion of statistical theory. The fullest characteristic of such a process

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*Analogous pattern can be constructed for the phase of the reflected signal; however, it represents significantly less interest and is not used at all in radar practice.

which at the same time is necessary in resolving a whole series of problems of the analysis and synthesis of radar systems, is a totality of multidimensional probability distributions of the values of this process. With a tendency of the dimension of the number to infinity there can be obtained the so-called functional of the probability density representing the probability density of the realizations of the process (see Section 1.4).

To directly find the distribution of probabilities proceeding from the real properties of target and its motion is very difficult. It appears unavoidable, in connection with this to use a more or less simplified radar model of the target. As such a model the presentation is very convenient of the target in the form of a totality of a large number of independently and randomly (in accordance with a definite distributive law) located reflecting elements, transferred one relative to another and relative to the ra set. In virtue of a central limiting theorem [20] the signal from such a totality of the reflecting elements is a normal random process. All multidimensional distributions of such a process are Gaussian and have the form

$$P(x_1,\ldots,x_n)=\frac{\exp\left\{-\frac{1}{2}\sum_{j,k=1}^{n} W_{jk}x_jx_k\right\}}{(2\pi)^{n/2}\sqrt{|R_{jk}|}} \qquad (1.3.2)$$

Here $x_j = x(t_j)\ (j=1,\ldots,n)$ is the totality of values of the process x(t) at the considered moment of time; $|R_{jk}|$ is the determinant of a correlated matrix composed of values of the function of correlation $R(t_j, t_k)$ of the signal x(t); $\|W_{jk}\|$ is the inverse matrix of the correlation; its elements are determined by the equations:

$$\sum_{i=1}^{n} R_{ji}W_{ik}=\delta_{jk}, \qquad (1.3.3)$$

where δ_{jk} is the Kronecker delta symbol:

$$\delta_{jk}=\begin{cases}1 & j=k\\ 0 & j\neq k.\end{cases}$$

For detecting the elements of the inverse matrix it is convenient to use the equality [18]

$$W_{jk}=\frac{\langle R_{kj}\rangle}{|R_{jk}|},$$

where $\langle R_{jk}\rangle$ is the cofactor of element R_{jk} of the correlation matrix.

In formula (1.3.2) the mathematical expectation of the reflected signal x(t) is assumed as equal to zero. This is observed in the case of a small change in the average density of reflectors at an interval equal to the wavelength when for the phase of $\varphi(t)$ it is natural to obtain in the interval of $(0,2\pi)$ a uniform distribution of probabilities.

Use of described model of radar target is justified by the fact that in a large number of cases the target has dimensions significantly exceeding the wavelength due to the fact that in calculating the signal reflected from such a target it is possible to use the geometric optics approach. Taking this into account, the signal from the target is considered as the sum of signals from separate "bright points" on the surface of the target. The number of these points for targets is usually very high, and their mutual location changes in a very indefinite manner depending upon the aspect of the target so that it can be fully considered random. Results of experimental determination of a one-dimensional distributive law for a signal reflected, for example, from a flying aircraft showed that this law sufficiently accurately approximates the normal. Width of the spectrum of fluctuations of the echo signal has an order of ten cycles per second [19].

The assumption of normality of the reflected radar signal is still justified to a large degree in the case of interfering reflections from the earth's surface, the clouds of metallized dipoles, and similar objects, since in these cases the number of randomly located elementary reflectors forming the interfering signal is very large.

In order to completely characterize the normal random process which we will consider a reflected signal, it is sufficient to find its function of correlation.

Let us calculate the function of correlation for an arbitrary totality of reflecting objects, and then specify the obtained result for a useful signal and passive interferences (interfering reflections). The calculation of the function of correlation of the signal, reflected from passive interference, for a number of particular cases is made in [1, 9, 20]. The calculation below of the function of correlation of the reflected signal will be made in a more general form so that the obtained results are useful for a large number of possible cases.

Let us consider at first the cases of single-frequency operation. A signal, reflected from every elementary reflecting object, is delayed at the time of propagation of the main signal with a modified amplitude and phase, and can be recorded in the form

$$x_i(t)=\sqrt{\frac{2P_0\lambda^2G_1G_2\sigma_i}{(4\pi)^3}}\,\mathrm{Re}\,g(\varphi_i,\theta_i,t)\times$$

$$\times s_i(t)\frac{u\left(t-\frac{2d_i}{c}\right)}{d_i^2}e^{i\omega_0\left(t-\frac{2d_i}{c}\right)}, \qquad (1.3.4)$$

where d_i, φ_i, θ_i are the spherical coordinates of the considered reflecting object, taken from the antenna of the radar set;

P_0 is the emitted power;

ω_0 is the carrier frequency;

λ is the wavelength;

G_1, G_2 are the gains of the transmitting and receiving antennas;

σ_i is the effective reflecting surface of object, neutralized along all possible orientations;

$g(\varphi, \theta, t)$ is the product of standardized coefficients of the directivity of the transmitting and receiving antennas.

Substituting into expression (1.3.4) the value of these functions at the same moment of time, we will disregard the displacement of the receiving diagram during the time of propagation of the signal.

The coefficient $s_1(t)$ takes into account the reflecting properties of the

object at the considered frequency at a given moment, $|s_l(t)|^2$ the relation of the effective reflecting surface to its mean value, and $s_l(t)$ characterizes the phase shift during reflection. This coefficient takes into account also the polarization properties of the reflecting object and is different for antenna systems with a different polarization. The change of $s_l(t)$ during the time of propagation was disregarded.

The signal from the totality of reflectors x(t) represents the sum of the signals $x_l(t)$ all along l. The upper limit of this sum (the number of reflectors) N can also be a random function of time (for example in examining the signal reflected from a strongly intersected site or the signal reflected from a cloud of dipoles when formulation of interference during observation is continued). However, usually during the time of correlation of the reflected signal N changes little, and this change practically does not affect the results. Therefore, we will not consider it henceforth.

Considering the above marked equality as a zero mean value of the reflected signal, and considering the position of the reflectors as independent, for the function of correlation $R(t_1, t_2)$ we obtain

$$R(t_1, t_2) = \mathrm{Re}\sum_{l=1}^{N} \overline{P_l g(\varphi_l, \theta_l, t_1) g^*(\varphi_l', \theta_l', t_2) s_l(t_1) s_l^*(t_2) \times \frac{u\left(t_1 - \frac{2d_l}{c}\right) u^*\left(t_2 - \frac{2d_l'}{c}\right)}{d_l^2 d_l'^2} e^{i\omega_0(t_1 - t_2) - i\frac{2\omega_0}{c}(d_l - d_l')}}. \quad (1.3.5) \qquad (1.3.5)$$

where

$$P_l = \frac{2P_0\lambda^2 G_1 G_2 \sigma_l}{(4\pi)^3};$$

d_l, φ_l, θ_l are the coordinates of the reflector at moment t_1, and $d_l', \varphi_l', \theta_l'$ at moment t_2.

Averaging in (1.3.5) should be produced by the random variables $d_l, d_l', \varphi_l, \varphi_l', \theta_l, \theta_l', s_l(t_1) s_l^*(t_2)$.

In order to obtain a simpler expression for $R(t_1, t_2)$, let us apply the following additional conditions, usually done in practice:

1. Let us disregard the change of angular coordinates during the time of correlation of the reflected signal, considering that during that time the reflector is displaced by a small share of the width of the pattern of directivity.

2. Let us segregate and consider separately two components of the motion of every reflector: one, that connected with the motion of all the totality of reflectors on the whole, and the other, that connected with the random transferences of reflectors. We will consider that the regular motion occurs with an unaltering radial speed of $v(\varphi,\theta)$ during the time of correlation.
With this

$$d_l^{*}=d_l-v(\varphi_l,\ \theta_l)(t_2-t_1)+\Delta d_l,$$

where Δd_l is the random transference during the time of $\Delta t=t_2-t_1$.

3. The random transferences and changes in the orientation of the reflector will be considered independent of its initial position. Moreover, the relative change of $d_{\underline{1}}$ during the time of correlation of the reflected signal will be considered low.

4. Differences in the regular radial transferences of reflectors striking the antenna beam will be considered small as compared with the interval of solution with respect to range. This will allow us to substitute into the argument $u\left(t_2-\frac{2d_l^{*}}{c}\right)$ the magnitude $d_l^{*}=d_l-v_0(t_2-t_1)$,
where v_0 is the mean value of velocity for all reflectors $v(\varphi,\theta)$.

5. The properties of all reflectors will be considered identical.

With the assumptions made, averaging into (1.3.5) should be done by $d_l, \varphi_l, \theta_l, \Delta d_l$ and by $s_l(t_1), s_l^{*}(t_2)$. Averaging by d_l, φ_l, θ_l reduces to multiplication of the sum by the probability density $P(d,\ \varphi,\ \theta)$ and to the integration all along the space. Moreover, in virtue of the identity of the terms under the integral there will enter the magnitude $NP(d,\varphi,\theta)=n(d,\varphi,\theta)$ representing the mean density of the reflectors at the point $(d,\ \varphi,\ \theta)$ at the moment of time t_1. As a result we

obtain

$$R(t_1,t_2)=\operatorname{Re} P_1\int_0^{\infty}dd\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\sin\theta\,\frac{n(d,\varphi,\theta)}{d^2}g(\varphi,\theta,t_1)\,g^*(\varphi,\theta,t_2)\times$$
$$\times u\left(t_1-\frac{2d}{c}\right)u^*\left(t_2-\frac{2d}{c}+\frac{2v_0}{c}(t_2-t_1)\right)e^{-i\omega_0\frac{2v(\varphi,\theta)}{c}(t_2-t_1)}\times$$
$$\times\overline{s(t_1)s^*(t_2)\,e^{i\omega_0\frac{2\Delta d}{c}}}. \qquad (1.3.6)$$

If in the considered totality there enter reflectors of various forms, then all the results will remain correct, but (1.3.5) and (1.3.6) should be summed through all the varieties of reflectors.

In the obtained formula it is convenient to join all members, independent of the law of modulation and determined by the character of fluctuations, conditioned by motion of the reflectors relative to the radar set and the rotation of the antenna. With this (1.3.6) is rewritten in the form

$$R(t_1,t_2)=\operatorname{Re} e^{i\omega_0(t_1-t_2)}\int_0^{\infty}R_0(z;t_1,t_2)\,u(t_1-z)\times$$
$$\times u^*\left[t_2-z+\frac{2v_0}{c}(t_2-t_1)\right]dz,$$

where

$$R_0(z;t_1,t_2)=\frac{2P_1}{cz^2}\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\,n(z,\varphi,\theta)\,g(\varphi,\theta,t_1)\times$$
$$\times g^*(\varphi,\theta,t_2)\,e^{-\frac{2v(\varphi,\theta)}{c}(t_2-t_1)}\,\overline{s(t_1)s^*(t_2)\,e^{i\frac{2\Delta d}{c}}}.$$

In the expression for $R_0(z;t_1,t_2)$ it is convenient to separate a factor [we will designate it by $\sigma(z)$], determining the distribution of the intensity of the reflected signal by the values of lag z, the coefficient of correlation of fluctuation $r(z;t_1,t_2)$ [considering that $r(z;t_1,t_1)=1$], which in a general case can also depend on z, and the factor $\exp\{i\omega_{\Sigma}(z)(t_2-t_1)\}$, the index of which characterizes the magnitude of the Doppler phase shift of the reflected signal corresponding to the interval of lags $(z, z+dz)$. Taking into account these designations, instead of (1.3.6) we obtain

$$R(t_1,t_2)=\operatorname{Re}\int_0^{\infty}\sigma(z)\,r(z;t_1,t_2)\,e^{i\omega_{\Sigma}(z)(t_2-t_1)}\times$$
$$\times u(t_1-z)\,u^*\left[t_2-z+\frac{2v_0}{c}(t_2-t_1)\right]e^{i\omega_0(t_1-t_2)}dz. \qquad (1.3.7)$$

Presentation (1.3.7) of the function of correlation of the reflected signal is very general and correct, in particular, at noncoinciding positions of the transmitting and receiving antennas. In the just now considered case of coinciding antennas the coefficient of correlation r and the Doppler shift $\omega_{д}$ do not depend on z.

The coefficient of correlation $r(z;\ t_1,\ t_2)$ takes into account the dependence of the fluctuations of the reflected signal on random and regular dislocations of reflectors and changes of their orientation relative to the radar set. Using (1.3.7), it is possible to specify the form of the function $r(z;\ t_1,\ t_2)$ for various particular cases of the distribution of reflectors in space (targets of different construction, a three-dimensional cloud of dipoles, terrestrial surface, etc.) and the law of motion of the antenna.

Formula (1.3.7) can be generalized without much trouble in the case of multifrequency emission with the arbitrary laws of modulation on each of the carrier frequencies. Repeating for that case the conclusion of formula, (1.3.7) and considering for simplicity the reflecting properties of objects identical for all utilized frequencies, we obtain

$$\begin{gathered}R(t_1,\ t_2)=\sum_{j,\,k}\int_0^\infty \sigma(z)\,\dot{r}_{jk}(z;\ t_1,\ t_2)\,e^{-i(\omega_j-\omega_k)z}\times\\ \times u_j(t_1-z)\,u_k^*\left(t_2-z+\frac{2v_0}{c}(t_2-t_1)\right)dz.\end{gathered} \tag{1.3.8}$$

where $u_j(t)$ is the law of modulation of oscillation with the carrier frequency ω_j. Usually in practice with multifrequency work the utilized frequencies are spaced sufficiently far apart, so that the corresponding reflected signals are statistically independent. Such a selection of the working frequencies allows to receive a gain in range of detection of the target and to improve the quality of selection of a moving target on a background of passive interferences (Chapter 4). Formula (1.3.8) allows to formulate the conditions of independence of the reflected signals having a large practical value.

As can be seen from (1.3.8), for the conversion of any component from $j \neq k$ to zero is sufficient, so that the wavelength of the separation frequency $\lambda_{jk} = \frac{2\pi}{|\omega_j - \omega_k|}$ was small as compared with the interval of noticeable change $\sigma(z) r(z, t_1, t_2)$ and the separation frequency $|\omega_j - \omega_k|$ was small as compared with the sum of the widths of the spectra of modulation of j-th and k-th signals. In using continuous emission without modulation $u_j(t) \equiv u_k(t) \equiv 1$, and the degree of correlation of signals is completely determined by the relationship between λ_{jk} and the extent of totality of reflecting objects with respect to range. The same is obtained in the case of a small dimension of the source of reflection as compared with the intervals of solution in range, corresponding to the considered signals. Moreover, $u_j\left(t_1 - \frac{2d}{c}\right) u_k^*\left(t_2 - \frac{2d}{c} + \frac{2v_0}{c}(t_2 - t_1)\right)$ can be carried out in (1.3.8) after the integral sign.

To the condition of independence of the reflected signals can be added the following graphic formulation: the reflected signals are statistically independent in the case where the wavelength corresponding to the separation frequency, is small compared with the extent of the considered reflecting object in range (with a given aspect of it relative to the radar set) or with the extent of intervals of resolution, corresponding to the laws of modulation of the considered signals.

Let us define concretely the obtained general relationships for the case of signals reflected from the target and interfering with the reflections. The dimensions of target we will henceforth assume small as compared with the width of the pattern of directivity of the antenna, as well as compared with the extent of the intervals of solution in range, and we will disregard the diffusion of modulation of the reflected signal owing to the extent of the target. Furthermore, in most cases we will disregard the distortion of the modulation of the reflected signal owing to the Doppler effect.

The function of the correlation of the signal reflected from the target is recorded in the form

$$R_c(t_1,t_2)=\mathrm{Re}\,P_c\,g(\varphi_c,\theta_c,t_1)\,g^*(\varphi_c,\theta_c,t_2)\,u(t_1-\tau)\,u^*(t_2-\tau)\times$$
$$\times\rho(t_1-t_2)\,e^{i(\omega_0+\omega_{\text{Д}})(t_1-t_2)}, \qquad (1.3.9)$$

where $\rho(t_1-t_2)$ is the coefficient of the correlation of fluctuations of the reflected signal;

P_c is the power.

The factor $g(\varphi_c,\theta_c,t)$ in resolving a number of problems can be included in the law of modulation of the reflected signal $u(t-\tau)$.

In this case the function of correlation during multifrequency emission will be converted in a similar manner

$$R_c(t_1,t_2)=\mathrm{Re}\sum_{j,k}\frac{A_jA_k}{2}\,g(\varphi_c,\theta_c,t_1)\,g^*(\varphi_c,\theta_c,t_2)\times$$
$$\times u_j(t_1-\tau)\,u_k^*(t_2-\tau)\,\rho_{jk}\,e^{i(\omega_j t_1-\omega_k t_2+\omega_{\text{Д}j}t_1-\omega_{\text{Д}k}t_2)}\rho(t_1-t_2), \qquad (1.3.10)$$

where $\rho(t_1-t_2)$ is the coefficient of the correlation of fluctuations (exemplarily assumed as identical to all the utilized frequencies);

$\omega_{\text{Д}j}$ is the Doppler shift of the j-th carrier frequency;

A_j is the amplitude of the j-th reflected signal.

The coefficient ρ_{jk} determines the degree of correlation of the j-th and k-th signals

$$\rho_{jk}=\int_{-\infty}^{\infty}\sigma_c(r)\,e^{-i(\omega_j-\omega_k)\frac{2r}{c}}\,dr, \qquad (1.3.11)$$

where $\sigma_c(r)$ is the distribution density of the reflecting surface.

For interfering reflections from small-size objects (henceforth we will call such objects discrete interferences or interferences of the type "interfering, false target") the function of correlation of the reflected signal can also be recorded in the form of (1.3.9), (1.3.10).

There is significant interest in the case of a slow change of the function $f(z)=\sigma(z)r(z;t_1,t_2)\,e^{i\omega_д(z)(t_1-t_2)}$, corresponding to extensive passive interference. Physically it is obvious that this case is the most complicated from the viewpoint of protection of the radar set from the influence of interfering reflections. Therefore, an analysis and synthesis of the means of protection from the interferences under consideration in this case correspond to the minimum approach ensuring the best effect in the worst case. The practical use of the results of synthesis does not require a knowledge of the distribution of elementary reflectors in space which profitably distinguishes the given case from the remaining.

In the case of extensive interference, the expression for the function of correlation can be simplified considerably. If the main signal is a multiple modulated single sending and f(z) changes little in the interval equal to the duration of the sending then from (1.3.7) we obtain

$$R_п(t_1,t_2)\approx \mathrm{Re}\,\sigma(t_1)\,r(t_1;t_1,t_2)\,T_{эф}C_1[\alpha(t_1-t_2)]\,e^{i(\omega_0+\omega_{дп})(t_1-t_2)}, \tag{1.3.12}$$

where $\omega_{дп}$ is the Doppler frequency of interference;

$C(\tau)$ is determined by the equality (1.2.2) $\alpha=1+\frac{2v_r}{c}$.

The factor α in this formula takes into account the distortion of modulation owing to the motion of the target. Usually they are sufficiently weak and their consequence need not be considered. Calculation of these distortions will be given in Section 4.9.

If u(t) is the periodic signal, then for obtaining a formula, analogous to (1.3.12), it is sufficient to require a small change of f(t) after the duration of the period, and in the case of stationary noise modulation of the main signal — during the time of, and several times exceeding the time of correlation of the modulating random process. Moreover, in accordance with (1.3.7) and (1.2.2')

$$R_п(t_1,t_2)\approx \mathrm{Re}\,C[\alpha(t_1-t_2)]\times$$
$$\times e^{i\omega_0(t_1-t_2)}\int_0^\infty \sigma(z)\,r(z;t_1,t_2)\,e^{i\omega_д(z)(t_1-t_2)}\,dz. \tag{1.3.13}$$

As can be seen from (1.3.12) and (1.3.13), in the case of extensive passive interference the transience is dependent only on the presence of the factor $r(z; t_1, t_2)$. Returning to formula (1.3.5), it is easy to perceive that $r(z; t_1, t_2)$ depends not only on the difference of t_1 and t_2, but also on their absolute values which are connected with the change of position of the pattern of directivity. In the mode of detection the position of the pattern of directivity usually changes sufficiently slowly. During interference extended at angles, $r(z; t_1, t_2)$ can be considered as the slow function of t_1, which diminishes sufficiently fast with an increase of $|t_1 - t_2|$. Henceforth, we will consider this assumption to be carried out. Moreover, as follows from (1.3.12) and (1.3.13), the signal reflected from extensive passive interference can be considered a quasi-stationary random process. In Chapter 4 it will be shown that for use in problems of analysis and synthesis of results obtained for stationary interference, it is sufficient to require a smallness of change of f(z) in the interval of solution in range.

In the case of extensive passive interference and multifrequency emission the above mentioned conditions of the independence of reflected signals, corresponding to various carrier frequencies, coincide with the conditions with which the separate reception of these signals is possible (1.2) and are always satisfied. Therefore, the function of correlation of the reflected signal in this case is the sum of the functions of correlation of the separate constituents.

It was earlier indicated that the normality of the reflected radar signal is based on the presentation of the target in the form of a totality of a large number of random, transferred one relative to another, "brilliant" points. The parameters of a normal random process utilized for a description of this signal depend, naturally, on the aspect of the target since with change in its orientation relative to the radar set the quantity and peculiarity changes of those "brilliant" points, which participate in forming the echo signals. In most cases the target (for example an aircraft) for the duration of a long interval of time does not practically

changes its orientation so that the presentation of the reflected signal in the form of a normal random process with fixed values of parameters is comprehensive. In some cases the target in this sense is not stabilized and participates in complicated rotations (for example artificial earth satellites). Moreover, the reflected signal for every orientation of a target, as before, can be considered normal, but the orientation of the target itself is random.

Usually the rotation of a target is rather slow, and in a number of cases after intervals of time the parameters interesting us of the normal random process (echo signal) practically do not change, although they remain indefinite. In connection with this in calculating the characteristics of radar systems working on such ballistic targets it is possible, as before, to use an idealized reflected signal in the form of a normal random process neutralizing then the calculated characteristics of the radar set, taking into account the probability of the various target orientations. With this neutralizing, if the form of the function of correlation of the echo signal practically does not depend on the orientation of the target it is sufficient to limit ourselves to those of its characteristics as the probability of the various magnitudes of the effective reflecting surface which determines the intensity of the reflected signal.

The above conducted investigation of the characteristics of the reflected radar signal is connected with those of its changes which are determined by the process of reflection from the target. With an incoherent pulse signal there are still additional random changes of the echo signal connected with "jumps" of the initial phase of the high-frequency filling of adjacent pulses of the main signal. In connection with this the distribution of probabilities for an incoherent echo signal, considering its change from pulse to pulse, is not normal. This question will be considered in more detail in the chapters devoted to the calculation of the characteristics of incoherent radar systems.

1.4. The Received Radar Signal

The input signal for different radar systems (the received signal) is a mixture (sum) of the reflected signal and set noises of the receiver, converted in the input of the system. Into the received signal one should also include various interferences inevitable under real conditions of the use of radar systems.

Earlier it was indicated that in many cases the reflected radar signal is subordinated to the normal distribution of probabilities. Set noises, as it is known, in most cases can also be considered as normal and "white" (having constant spectral density of N_0 within the limits of the band pass of the receiver); also normal are many forms of interfering signals including such wide-spread forms as active noise interferences and reflections from passive interferences in the form of a cloud of dipole reflectors and also reflections from the earth's and sea's surface. In virtue of the circumstances presented, the received radar signal is frequently a normal random process. Its mathematical description is given as also for a reflected signal, by the multi-dimensional probability of density of the form (1.3.2).

In distinction from the reflected signal the correlated function of the received signal clearly determining the distributive law of probabilities, is the sum of the correlated functions of all constituents of the signal. Henceforth, in the mathematical description of a received signal y(t), besides the multi-dimensional probability density, there will be widely used the functional of the probability density which is obtained from the n-dimensional probability density for the values of the signal at moments $t_j = j\Delta t$ $(j = 1, 2, \ldots, n)$ as a result of maximum transition during $\Delta t \to 0$ and $n \to \infty$. Let us consider more specifically this functional and its properties.

Limit of the index of exponents in (1.3.2) is easily calculated

$$\lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \left[-\frac{1}{2} \sum_{j,k=1}^{n} W_{jk} y_j y_k \right] = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \left[-\frac{1}{2} \sum_{j,k=1}^{n} \frac{W_{jk}}{\Delta t^2} y_j y_k \Delta t^2 \right] =$$

$$= -\frac{1}{2} \int_0^T \int_0^T W(t_1, t_2) y(t_1) y(t_2) \, dt_1 dt_2,$$

where

$$y_j = y(j\Delta t) \quad \text{and} \quad W(t_1, t_2) = \lim_{\substack{j\Delta t \to t_1 \\ k\Delta t \to t_2}} \frac{W(j\Delta t, k\Delta t)}{\Delta t^2}.$$

The functional of the density of probabilities is recorded thus:

$$F[y(t)] = K_0 \exp\left\{ -\frac{1}{2} \int_0^T \int_0^T W(t_1; t_2) y(t_1) y(t_2) \, dt_1 dt_2 \right\}, \tag{1.4.1}$$

Equation (1.3.3), determining the elements of the inverse correlated matrix, can be written in the form

$$\sum_{l=1}^{n} R_{jl} \frac{W_{lk}}{\Delta t^2} \Delta t = \frac{\delta_{jk}}{\Delta t}.$$

converted at $\Delta t \to 0$ and $n \to \infty$ into an integral equation

$$\int_0^T R(t_1, s) W(s, t_2) \, ds = \delta(t_1 - t_2), \tag{1.4.2}$$

where $R(t_1, t_2)$ is the function of correlation of the received signal, equal to

$$R(t_1, t_2) = P_c \operatorname{Re} u(t_1 - \tau) u(t_2 - \tau) \rho(t_1 - t_2) \times$$
$$\times e^{i(\omega_0 + \omega_д)(t_1 - t_2)} + N_0 \delta(t_1 - t_2). \tag{1.4.3}$$

Here N_0 is the spectral density of set noises of the receiving mechanism.

The factor $K_0 = \lim_{n \to \infty} \frac{1}{(2\pi)^{n/2} \sqrt{|R_{jk}|}}$ in the expression (1.4.1), in general, is infinitely large. However, in resolving practical problems, this circumstance is immaterial since in the synthesis of measuring radar systems we use (Chapter 6) the logarithmic derivative of the functional of probability density, and in synthesis of radar systems of detection (Chapter 3) -- the relation of these

functionals. It is easy to verify that both these functions are finite.

The logarithmic derivative of the functional F[y(t)] by the arbitrary parameter λ is determined as the sum of logarithmic derivative K_0 and the derivative of the index of exponents in (1.4.1) by this parameter, which physically can be range, velocity or any other coordinate of the target.

For $\frac{\partial \ln |R_{jk}|}{\partial\lambda}$ we have

$$\frac{\partial \ln |R_{jk}|}{\partial\lambda} = \frac{\frac{\partial}{\partial\lambda}|R_{jk}|}{|R_{jk}|} =$$

$$= \frac{1}{|R_{jk}|}\left\{\begin{vmatrix} R'_{11}R_{12}\ldots R_{1n} \\ \ldots\ldots\ldots \\ R'_{n1}R_{n2}\ldots R_{nn}\end{vmatrix} + \begin{vmatrix} R_{11}R'_{12}\ldots R_{1n} \\ \ldots\ldots\ldots \\ R_{n1}R'_{n2}\ldots R_{nn}\end{vmatrix} + \ldots +\right.$$

$$\left. + \begin{vmatrix} R_{11}R_{12}\ldots R'_{1n} \\ \ldots\ldots\ldots \\ R_{n1}R_{n2}\ldots R'_{nn}\end{vmatrix}\right\} = \frac{1}{|R_{jk}|}\left\{\sum_{j=1}^{n} R'_{j1}(R_{j1}) +\right.$$

$$\left. + \sum_{j=1}^{n} R'_{j2}(R_{j2}) + \ldots + \sum_{j=1}^{n} R'_{jn}(R_{jn})\right\} =$$

$$= \sum_{j,k=1}^{n} R'_{jk}\frac{(R_{jk})}{|R_{jk}|} = \sum_{j,k=1}^{n} R'_{jk}\frac{W_{jk}}{\Delta t^2}\Delta t^2. \qquad (1.4.4)$$

Here (R_{jk}) is the cofactor of the element R_{jk}; the dash signifies the differentiation on the parameter λ.

As a result of maximum transition at $n \to \infty$ and $\Delta t \to 0$ we will obtain

$$\frac{\partial \ln K_0}{\partial\lambda} = -\frac{1}{2}\lim_{\substack{\Delta t\to 0\\ n\to\infty}}\frac{\partial \ln |R_{jk}|}{\partial\lambda} = -\frac{1}{2}\int_0^T\int_0^T\frac{\partial R(t_1, t_2)}{\partial\lambda}\times$$
$$\times W(t_1, t_2)\,dt_1\,dt_2, \qquad (1.4.4')$$

where the function $W(t_1, t_2)$ is determined by equation (1.4.2).

Finally, for the derivative on the parameter λ from the logarithum of the functional F[y(t)] we have

$$\frac{\partial \ln F[y(t)]}{\partial\lambda} = -\frac{1}{2}\int_0^T\int_0^T\frac{\partial R(t_1, t_2)}{\partial\lambda}W(t_1, t_2)\,dt_1\,dt_2 -$$
$$-\frac{1}{2}\int_0^T\int_0^T\frac{\partial W(t_1, t_2)}{\partial\lambda}y(t_1)\,y(t_2)\,dt_1\,dt_2. \qquad (1.4.5)$$

Formula (1.4.4) easily allows to establish the finiteness of the relation of determinants of correlated matrices for two arbitrary normal processes. In the particular case interesting us, pertaining to the problems of detection, one of the considered processes is the arbitrary interference with the function of correlation $R_{\text{п}}(t_1, t_2)$ and second is the sum of the useful signal and the interference with the function of correlation, which can be recorded in the form $\lambda R_{c}(t_1, t_2) + R_{\text{п}}(t_1, t_2)$. Moreover, we are interested in the determinant

$$|B_{jk}(\lambda)| = \frac{|R_{\text{п}jk}|}{|\lambda R_{cjk} + R_{\text{п}jk}|} = \frac{1}{|\lambda R_{cjk} + R_{\text{п}jk}|\,|W_{\text{п}jk}|} = \\ = |\delta_{jk} + \lambda Q_{jk}\Delta t|^{-1}, \qquad (1.4.6)$$

where $\|W_{\text{п}jk}\|$ is the matrix, the inverse of $\|R_{\text{п}jk}\|$; and

$$Q_{jk} = \sum_{l=1}^{n} Rc_{jl} \frac{W_{\text{п}lk}}{\Delta t^2} \Delta t.$$

In the limit at $\Delta t \to 0$ and $n \to \infty$ we obtain

$$Q(t_1, t_2) = \lim_{\substack{j\Delta t \to t_1 \\ k\Delta t \to t_2}} Q_{jk} = \int_0^T Rc(t_1, s)\, W_{\text{п}}(s, t_2)\, ds. \qquad (1.4.7)$$

Applying to the determinant $|B_{jk}(\lambda)|^{-1}$ the formula

$$|R_{jk}| = \exp\left\{\int_0^{\lambda} d\lambda \sum_{j,k=1}^{n} R'_{jk} \frac{\Gamma'_{jk}}{\Delta t^2} \Delta t^2\right\}, \qquad (1.4.8)$$

which follows from (1.4.4), where

$$\sum_{l=1}^{n} R_{jl} W_{lk} = \delta_{jk},$$

we have

$$|B_{jk}(\lambda)|^{-1} = \exp\left\{\int_0^{\lambda} d\lambda \sum_{j,k=1}^{n} Q_{jk} \frac{W_{lk}}{\Delta t} \Delta t^2\right\},$$

whereby for W_{jk} the equation is correct

$$\sum_{l=1}^{n} |\delta_{jl} + \lambda Q_{jl}\Delta t|\, W_{lk} = \delta_{jk}.$$

Changing to the limit at $\Delta t \to 0$ and $n \to \infty$, we have

$$B^{-1}(\lambda) = \lim_{\substack{j\Delta t \to t_1 \\ k\Delta t \to t_2}} |B_{jk}(\lambda)|^{-1} = \\ = \exp\left\{\int_0^{\lambda} d\lambda \int_0^T \int_0^T Q(t_1, t_2)\, V(t_1, t_2)\, dt_1\, dt_2\right\}. \qquad (1.4.9)$$

where $V(t_1, t_2)=\lim\limits_{\substack{j\Delta t\to t_1 \\ k\Delta t\to t_2}} \frac{V_{jk}}{\Delta t}$

and there is determined from the equation

$$\int_0^T [\lambda Q(t_1, s)+\delta(t_1-s)]V(s, t_2)\,ds=\delta(t_1-t_2), \qquad (1.4.10)$$

which can be rewritten in the form

$$\int_0^T Q(t_1, s)V(s, t_2)\,ds=\frac{1}{\lambda}[\delta(t_1-t_2)-V(t_1, t_2)]=\theta(t_1, t_2). \qquad (1.4.11)$$

Comparing (1.4.9) and (1.4.11) and considering the symmetry of the matrix $\|Q_{jk}\|$, we see that

$$B(\lambda)=\exp\left\{-\int_0^\lambda d\lambda\int_0^T \theta(t, t)\,dt\right\}, \qquad (1.4.12)$$

and the function $\theta(t_1, t_2)$ in accordance with (1.4.10) and 1.4.11) is determined by the integral equation

$$\theta(t_1, t_2)+\lambda\int_0^T Q(t_1, s)\theta(s, t_2)\,ds=Q(t_1, t_2). \qquad (1.4.13)$$

Formulas (1.4.12) and (1.4.13) allow to find the limit of the determinant of the matrix of the form (1.4.6). Thus, the relation of functionals of the probability of density of the received signal with the absence and presence of the useful signal is obtained by the application of the obtained formulas.

All the expressions connected with the functional of the density of probabilities are easily generalized by a multi-dimensional case where for some resolutions of the parameters of the target it is necessary to use m signals received on independent channels. These problems will be considered in Chapter 10 and 12, devoted to multi-dimensional measurements.

1.5. Jamming In Radar

Under the conditions of the practical use of various radar systems an essential role is played by their noiseproof feature, which in many cases predetermines the

possibility of fulfillment by radar of problems placed before it. Jamming in radar can be created naturally and artificially. Natural jamming includes, in the first place, set noises of the receiving mechanisms, and also reflections from local objects, the earth, and sea surfaces, galactic noises, and other factors, which are connected with the peculiarities of the construction of a radar set and the conditions of its operation. Here touch the so-called mutual interferences connected with the presence in space of the simultaneous emissions of many radar stations. The main forms of natural interferences and their characteristics were briefly described above with considerations of the properties of the reflected and received radar signals. Here will be cited that which is necessary for the further information on artificial interferences especially created for jamming the normal operation of radar stations [1, 21, 22, 23, 76].

Artificial, specially organized interferences by their own character can be active and passive. Active interferences are created by special jamming transmitters, passive are the result of the reflection of a main radar signal from specially fixed reflecting objects. Let us first consider active interferences.

According to target designation, active interferences can be divided into spot and barrage. Spot jammings are tuned to a fixed frequency, have a comparatively narrow spectrum and are intended for the suppression of separate radar sets. For the creation of such interference, preliminary and sufficiently accurate intelligence of the main parameters of the suppressed radar set is necessary in connection with which the corresponding jamming equipment should be joined with the reconnaissance receiver. This form of jamming is intended basically for individual radar protection of an object and allows to create the necessary excess of level of interference above the reflected radar signal even for stations with a small operating range. Barrage jammings cover a sufficiently wide range of frequencies, do not require accurate reconnaissance equipment and can simultaneously jam several radar stations. Barrage jamming can be used for individual protection

as well as for radar cover of a group of targets with the help of special producers of interference.

The important forms of active interferences are noise and reciprocal (relay). Interference in the form of continuous noise signal ("smooth" noise interference) is able to disturb the operation of range measuring mechanisms. For increasing the effectiveness of noise interference in reference to goniometrical radar systems it is necessary to increase the spectral constituents in the areas of possible scanning frequencies.

Henceforth, for noise interference there is taken an idealization in the form of normal white noise which is justified by the fact that the width of the spectrum of noise interference in most cases significantly exceeds the bandwidth of the input circuits of the receiving mechanism. In accordance with this idealization the noise interference will be characterized by a single parameter with a spectral density of N_n.

Reciprocal active interference controls the main radar signal: corresponding jamming transmitters either overradiate an intensive and, especially, a thoroughly modulated signal of a radar station, or are triggered by this signal. Reciprocal interference can have a pulse as well as a continuous character, and can be either single or repeated (with "multiplication"). Transmitters of single reciprocal interferences intended for jamming autotracking systems can give signals modulated in addition by amplitude or lag time. Multiple reciprocal (limitation) jamming is applied against radar detection sets. Such jamming creates false blips on the radar screen, which can mask a group of targets proceeding from behind, and a pulse repetition period selected in the appropriate way — and a producer of interferences. Transmitters of reciprocal interference are distinguished by simplicity, small delay time of relatively useful jamming signal, broad-bandness, small recovery time and, as a result, the ability to simultaneous jam many radar stations.

Besides noise and reciprocal jamming, one should note also pulse jamming with a regular period of repetition (interference of the type "railing") and random pulse jamming (RPJ) with a random repetition period. The amplitudes of pulses of interference will be considered constant, and their duration -- of the same order as the pulses of a useful signal. Considering the average porosity of random pulse jamming $\bar{Q}=\frac{1}{\nu\tau_{\text{и}}}$ where ν is the mean frequency of the appearance of interference pulses, significantly larger than one can be taken for the probability of appearance of n pulses of RPJ in the interval of duration τ of the Poisson distribution

$$P(n)=\frac{(\nu\tau)^n}{n!}\exp\{-\nu\tau\}. \qquad (1.5.1)$$

At large ν, when $\nu\tau_{\text{и}}$ is comparable with one, should consider that physically, pulses of RPJ cannot be overlapped. Then the full probability μdt of appearance of a pulse of interference in the interval of (t, t + dt) is equal to the product of the probability νdt of the appearance of a pulse in this interval under the condition that in the interval of $(t-\tau_{\text{и}}, t)$ the pulses of RPJ do not appear on the probability of this condition. It follows from this

$$\mu=\frac{\nu}{1+\nu\tau_{\text{и}}}. \qquad (1.5.2)$$

Therefore, in the case of large $\nu\tau_{\text{и}}$ it is possible to use distribution (1.5.1), taking instead of ν the equivalent mean frequency μ.

The described forms of active interferences can be created practically in any of the presently utilized bands of frequencies and act either constantly or intermittently at definite or random moments of time.

By passive interference we will mainly mean dipole reflectors from various materials. Reflectors are made up into packs which usually contain ten thousand dipoles. The dimension, form, and material of the dipole is determined by its basic characteristics: the neutralized effective reflecting surface, polarization of the signal reflected from it, the rate of descent, the coefficient of adhesion.

The effective reflecting surface of one half-wave dipole is estimated by the formula $0{,}18\lambda^2$ where λ is the working wavelength. The coefficient of adhesion estimates the decrease of the number of effective dipoles owing to adhesion and the breakdown of the reflectors with aperture of the pack. An important characteristic of passive interference is also the time of dispersion necessary for full "development" of a cloud of interference after aperture of the pack. The signal reflected from the cloud of dipole reflectors will be considered a normal random process. The form of its function of correlation is presented in Section 1.3.

Equipment for passive interference consists of automata for discharging the reflectors and also special equipment for raising the interference in front, on the side, and to the rear of the covered object.

An important form of interference is also the so-called false targets which fulfill a misinforming role and are in contrast to the method of coarse force (superpowerful transmitters of active interferences, "dense" clouds of dipoles of a large extension). In a number of cases, as a means of creating false targets, there can be used active and passive interferences -- transmitters of reciprocal (imitation) jamming, small separately located clouds of dipole reflectors, and so forth. Along with this there can be applied special equipment -- decoy ("radar traps"), which will carry special equipment simulating the echo signal from a target.

1.6. Conclusion

In this chapter we considered the main characteristics of various applied laws of modulation of the main radar signal, the fluctuating properties of the reflected signal, and also possible interferences.

Very important for the future is the presentation of the echo signal and, in a series of incidents, the entire signal received in the form of a normal random

process. Since this presentation corresponds well to the essence of the majority of problems found at present in radar, then it limits little the community of the below obtained results. Some of these results, as will be shown in subsequent chapters also pervade signals not subordinated, in general, to the normal distributive law of probabilities.

CHAPTER 2

THE INFLUENCE OF SIGNALS AND INTERFERENCES ON THE ELEMENTS OF THE RADIO RECEIVER MECHANISM

2.1. Introductory Remarks

The radio receiver mechanism is an obligatory component part of any radar set. Moreover, within the limits of one radar station the same receiver is frequently used to perform the most diverse functions connected with detection, the measurement of coordinates, and tracking of any target parameter.

The receiving mechanism, to a great degree, is subject to the influence of interferences of a distinct form and fluctuations of the signal reflected from the target, whereupon it will convert the signal and interference in such a way that their characteristics at the output appear to be significantly modified and dependent on the selection of parameters of the receiver. Therefore, during the analysis of systems of radar detection and measurement it is necessary to know the characteristics of signals and interferences at the output of the receiver or its separate elements.

The receiving mechanisms differ, depending upon the functions performed by them and the form of the received signal. Thus, there can be receivers of coherent and incoherent, pulse and continuous signals. Of course it is difficult to describe all the possible modifications of the circuits of receiving mechanisms. The circuits applied in various channels of the radar set will be given in subsequent chapters. However, in spite of the difference of forms of receivers many

elements from which they are constructed appear to be identical and change only in their requirements. We will dwell briefly on the elements of the receiver. They include the input high-frequency mechanisms including the waveguide channels and preselectors. Many modern receiving mechanisms have high frequency amplifers built on the basis of parametric amplifiers, molecular amplifiers, or traveling wave tube amplifiers. The obligatory elements of radar receivers are mixers, in which there occurs the conversion of the signal spectrum owing to the interaction of the signal and heterodyne voltages. The essential peculiarity of coherent signal receivers is the fact that the voltage of heterodynes is rigidly connected by phase with the main signal. Usually this is obtained by the fact that the heterodynes and the transmitter of the station have, as a general source, a quartz generator. Besides this, the application is possible of a coherent heterodyne phaseing from the transmitter. In incoherent reception such a rigid connection between the main signal and the heterodyne voltage does not exist and, as heterodynes, in this case, klystron generators are usually used.

The majority of modern receivers have intermediate frequency amplifiers (IFA). The exception constitutes receivers of a direct amplifier which in radar are applied comparatively rarely in view of their low sensitivity. In incoherent pulse receivers the pass band of the IPA is selected in accordance with the spectrum of received pulses (or wider). In optimum receivers of a pulse coherent signal, as one will see below, the IPA should consist at minimum of two amplifiers. The first amplifier is a wide band, and in it is carried out the gating of the signal. The second amplifier is a narrow band with the pass band coordinated with the spectrum of fluctuatious of the received signal.

After proceeding through the IPA the signal is detected by a second detector and proceeds to the input of the video amplifier in pulse incoherent receivers, or the low frequency amplifier in coherent receivers. From the output of these amplifiers the signal proceeds to various channels of the radar set, intended

for the detection of targets, range measurement, angles etc.

In radar sets using for the measurement of angles the principle of instantaneous amplitude comparison of signals, the receiving mechanism usually consists of three identical channels. Signals from the output IFA of these channels move to the phase detectors, in the output of which are produced voltages proportional to the angular displacement of the direction on target, relative to an equisignal direction. Phase detectors are also applied in the channels of measurement of velocity of a target of stations with coherent emission and in autofrequency control systems.

Finally, the majority of modern receiving mechanisms have a system of automatic gain control (AGC). For the various types of receiving mechanisms the system of AGC can be accomplished differently. However in investigating the influence of the AGC system on the characteristics of a signal, the receiving mechanism with AGC can be represented in the form of an equivalent diagram which is common for many cases.

It is advisable to preliminarily expound the problems of the action of signals and interferences on the enumerated elements of the radio receiving mechanism, inasmuch as they are common for the majority of subsequent chapters. Furthermore, it is useful to introduce definite idealizations for the characteristics of the elements of the receiving mechanism and to stipulate the limits of their application, in order to then widely use such idealized characteristics. These problems are the contents of the present chapter. Certainly, it in no way pretends to be a full account of all the problems of the influence of interferences on receiving mechanisms. Specialists in radio receiving mechanisms may remain discontented by the fact that in reference to the separate elements of receiver the idealizations taken are too rough. However, one should consider that frequently this is done for the purpose of simplifying the analysis of radar systems on the whole, which is conducted in subsequent chapters. Let us note that the material presented can

represent an independent interest under the influence of interferences jointly with a useful signal at the receiving mechanism.

In the account are widely used the results of analysis of the influence of random processes on the elements of radio reception mechanisms which are contained in the works of V. I. Bunimovich [24], B. R. Levin [17], S. O. Rays [25] and others. In view of the limits of volume, detailed mathematical computations are lowered, and only the final expressions are given and the method of obtaining them. Along with the already known results, in the chapter is contained the original material relating to the problems of the influence of random signals and interferences on the receiving mechanism with automatic gain control.

Before going to an analysis of the passage of signals and interferences through the separate elements of the receiving mechanism let us consider the problem of noises in the receiver.

2.2. Noises In Receiver. The Noise Factor and the Effective Noise Temperature of the Receiving Mechanism

Since noises are of a wider band than the receiving mechanism we will henceforth consider that in the input of the ideal receiver there takes place a noise voltage idealized by white noise. Then the only characteristic that will be necessary in subsequent chapters is the spectral density N_0 of this equivalent noise converted to the input of the receiver. In order to find this characteristic let us consider the cause of the appearance of noises and the quantitative characteristics of noise. First of all let us note that if even the receiving mechanism itself was ideally noiseless, in the input of the receiver there would take place a noise voltage. The causes of the appearance of these input noises will be indicated somewhat below. Since the receiving mechanism itself is imperfect and creates additional noises, then noise voltage in the output of the receiver will be determined by input as well as set noises. If the receiver does not contain low-noise high frequency amplifiers then the noise voltage determined in the

output will be set noises.

In order to quantitatively estimate how far a real receiver differes from an ideal noiseless set there is usually introduced the conception of the noise factor of a receiving mechanism.

The noise factor F of a certain linear quadripole is called the number showing how many times the ratio of signal to noise, by the power on its input, is larger than the corresponding ratio of signal to noise in the output,

$$F=\frac{\frac{P_{\text{с вх}}}{P_{\text{ш вх}}}}{\frac{P_{\text{с вых}}}{P_{\text{ш вых}}}}, \qquad (2.2.1)$$

where $\frac{P_{\text{с вх}}}{P_{\text{ш вх}}}$ – is the ratio of signal power to noise power at the input in the quadripole-pass band;

$\frac{P_{\text{с вых}}}{P_{\text{ш вых}}}$ – is the ratio of signal power to noise power on the output.

From the relationship (2.2.1) it is clear that for an ideal noiseless quadripole the noise factor is equal to one and for any real one $F>1$.

Let us represent the receiving mechanism in the form of series connected quadripoles which accordingly have the noise factors: F_1, F_2, F_3,... If the loads of the quadripoles are coordinated, then for the noise factor of the receiving mechanism it is simple to obtain the following relationship:

$$F=F_1+\frac{F_2-1}{G_1}+\frac{F_3-1}{G_1G_2}+\ldots \qquad (2.2.2)$$

where G_1 and G_2 are the amplification factors of quadripoles by power.

From the obtained expression it is clear that if the receiver has a high frequency amplifier with a large amplification factor, then its noise factor will be basically determined by the set noises of this amplifier and the input circuits.

However, receivers of the microwave range frequently do not have high frequency amplifiers. In such receivers the first intense noisy element will be

the mixer, the second--the intermediate frequency amplifier. Noises of the mixer are composed of set noises of the crystal detector and heterodyne noises. Usually the noise properties of the mixer are assumed to characterize the relative noise temperature

$$t_1 = \frac{T_1}{T} = G_1 F_1,$$

where T_1 is the effective noise temperature of the mixer;

T is the absolute temperature of the elements of the receiving mechanism;

G_1 and F_1 is the transmission factor by power and noise factor of mixer.

Then the noise factor of a receiver will be recorded in the form

$$F \approx \frac{t_1 + F_2 - 1}{G_1}. \qquad (2.2.3)$$

Here F_2 is the noise factor of the IFA.

Thus, the noise factor of the receiver is basically determined by noises of the intermediate frequency amplifier and noises of the mixer. There are many causes of appearance of noises of the IFA. It is possible to indicate, for example, such sources of noise as thermal resistance noises, noises appearing owing to the Schottky effect in electron tubes, and others.

As follows from formula (2.2.2), the noise factor of the whole intermediate frequency amplifier is basically determined by the noise factors of its first stages. Therefore, during projection of the receiving mechanisms, special attention is given to the noise properties of the input circuit and the first stages of the IFA. Not remaining in detail on these problems, we will indicate that the noise factor for microwave receivers in which there is no amplification at high frequency, usually occurs on the order of 10-15 db [27].

If the receiving mechanism has a high frequency amplifier, which is used as a traveling-wave tube (TW tube), then the noise factor of such a receiver has on the order of 3-5 db [28].

Knowing the magnitude of noise factor, it is possible to easily calculate the power of noise on the output of the IFA. From formula (2.2.1) we obtain

$$P_{\text{ш вых}} = GFP_{\text{ш вх}},$$

where G is the amplification factor by the power of the channel of the receiving mechanism to the second detector.

The power of noises on the input in the pass band of the IFA can be calculated by the known formula

$$P_{\text{ш вх}} \doteq kT_{\text{вх}}\Delta f_{\text{эф}}. \tag{2.2.4}$$

where $T_{\text{вх}}$ is the equivalent noise temperature in the input expressed in absolute units;

$\Delta f_{\text{эф}}$ is the effective pass band of the IFA;

k is the Boltzmann constant.

Then power of noise in the output of the IFA will be equal to

$$P_{\text{ш вых}} = kGFT_{\text{вх}}\Delta f_{\text{эф}} = kT_{\text{эф}}G\Delta f_{\text{эф}}. \tag{2.2.5}$$

where $T_{\text{эф}} = T_{\text{вх}}F$ is the effective noise temperature of the receiving mechanism.

It is necessary to note that the noise factor F included in these formulas is the noise factor measured at an effective noise temperature in the input $T_{\text{вх}}$, which can differ from the standard temperature of $T_0 = 290°$ K. Then one may use the following relationship between the noise factors measured at temperature $T_{\text{вх}}$, and the standard noise factor measured at temperature T_0,

$$F = 1 + \frac{T_0}{T_{\text{вх}}}(F_0 - 1). \tag{2.2.6}$$

Receiver noises limit the real sensitivity of the receiving mechanism and signify the maximum range of the radar station. Furthermore, owing to the presence of noises, there takes place additional fluctuating errors in the measurement of target position data. In connection with this, the most important problem

of projection of the receiving mechanisms of radar stations is lowering the level of noises.

Recently, to this end essential successes were reached owing mainly to the application of parametric and molecular amplifiers. Their set noises turn out to be comparably smaller than the level of input noises.

The input noises will be those noises appearing before the first low-noise amplifier. By the causes of their appearance, it is possible to them divide into two groups. To the first group belong noises which appear owing to the emission of a celestial background (cosmic noises), the secondary emission of an absorbant medium (atmospheric noises), thermal radiation of the earth sensed by the lateral lobes of the directivity pattern of the antenna. To the second group belong the noises which appear in the antenna and the elements of the receiving channel of the preceding amplifier. To them belong the noises which appear owing to the finite conductance of the surface of the metallic antenna, losses in the waveguide channel from the antenna to the low-noise amplifier, direct losses in the antenna switch etc.

If the constituent noises of antennas, appearing owing to heated earth, can be influenced by lowering the level of the lateral lobes owing to an improvement in the construction of the antenna, then more complicated is the problem of lowering the level of emission noise of the sky, received by the station from the direction of the main lobe of the directivity pattern of the antenna. This noise consists of constituents due to scattering and absorption in the atmosphere, and also of noise emission arriving from space located beyond the limits of the ionosphere (cosmic noise). Although the problem of the dependence of the level of noises on the working frequency of the station is still insufficiently investigated, there is information giving the possibility to judge the fact that the level of cosmic noise is inversely proportional to the frequency. This is illustrated by Fig. 2.1, borrowed from [26]. In the figure is shown the dependence

of effective noise temperature of ideal antennas on the frequency. At higher frequencies (above 10,000 Mc.) atmospheric noises start to show up strongly which are increased with an increase in the working frequency of the station. Hence, in particular, there apparently exists a certain optimum range of working frequencies at which the noise temperature of the antenna is minimum. Furthermore, the given graph gives the possibility to estimate the magnitude of the noise temperature of the antenna $T_{\text{а}}$.

Noises of the elements of the receiving channel to the low-noise amplifier can be easily estimated. If there is a certain source with an equivalent noise temperature $T_{\text{а}}$ and we must calculate the effective noise temperature on the output of the passive quadripole with a transmission factor with the power $G_{\text{тр}}$, then it is possible to use the following formula:

$$T_{\text{вх}} = T_{\text{а}} G_{\text{тр}} + T_{\text{тр}} (1 - G_{\text{тр}}), \tag{2.2.7}$$

where $T_{\text{тр}}$ is the absolute temperature of a passive quadripole.

The influence of additional quadripole elements can easily be estimated by a series application of expressions of this type.

In the case of the application of parametric or molecular amplifiers a more convenient characteristic of the noise properties of the receiver is the effective noise temperature $T_{\text{эф}}$. It will be composed of the noise temperature at the input and the temperature of the set noises of the amplifier:

$$T_{\text{эф}} = T_{\text{вх}} + T_{\text{ус}}.$$

As a result, we will obtain that the effective noise temperature of the receiving mechanism can be estimated by the formula

$$T_{\text{эф}} = T_{\text{а}} G_{\text{тр}} + T_{\text{тр}} (1 - G_{\text{тр}}) + T_{\text{ус}}, \tag{2.2.8}$$

where $T_{\text{а}}$ is the equivalent noise temperature of the antenna;

$T_{\text{тр}}$ is the absolute temperature of the waveguide channel;

$G_{\text{тр}}$ is the transmission factor by the power of this channel;

T_{yc} is the noise temperature of a high frequency amplifier.

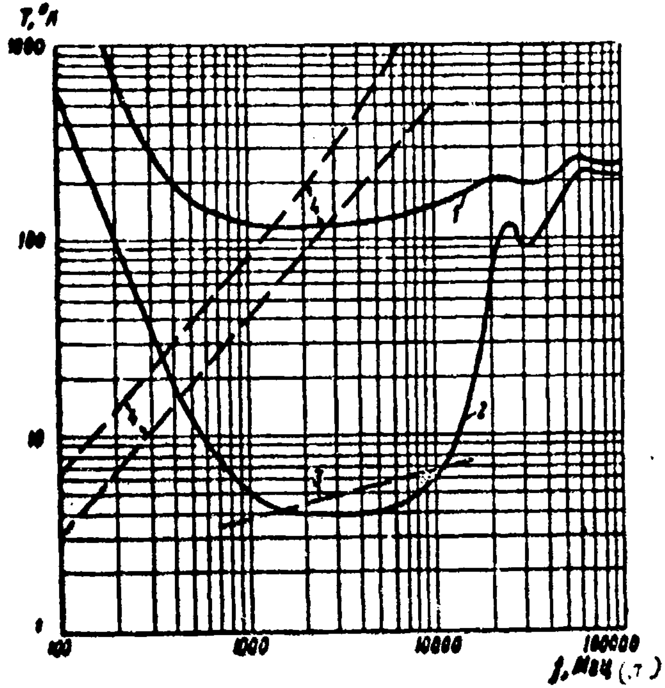

Fig. 2.1. Effective noise temperature of ideal antennas, molecular and parametric amplifiers: 1) ideal antenna aimed horizontally at galactic center; 2) ideal antenna vertically aimed at galactic pole; 3) molecular amplifier; 4) parametric amplifiers.
KEY: (a) f, Mc.

According to Fig. 2.1 it is possible to estimate the effective noise temperature of molecular and parametric amplifiers and its dependence on the working frequency. As can be seen from the graph, the noise temperature of the molecular amplifier turns out to be especially low (several degrees of K°); therefore, in receiving mechanisms with such amplifiers the input noises start to play a very large role. In connection with this, a serious problem is the decrease of noises on the input. This can be done by an improvement in the construction of the antenna, selection of the working frequency of the station, cooling of the elements of the antenna-waveguide channel to the low-noise amplifier and a decrease of losses in these elements.

Henceforth we will operate everywhere with the spectral density of noise N_0, which is easy to obtain from the above-mentioned relationships:

$$N_0 = \frac{1}{2} kFT \qquad (2.2.9)$$

or

$$N_0 = \frac{1}{2} k T_{\text{эф}}. \qquad (2.2.10)$$

2.3. Passage of Signal and Interferences Through an Intermediate Frequency Amplifier.

Let us consider the idealizations which will be taken in subsequent chapters with respect to the characteristics of the IFA and the question about how well these idealizations are performed in practice. While the signal on the input of any stage of the IFA is significantly less than the region of linearity of the anode-grid characteristic of the stage, the IFA can be simply characterized by its own amplification factor and frequency response.

Depending upon the construction of the intermediate frequency amplifier, the form of its frequency response can be different. If the required pass band of the receiver is comparatively small $\left(\frac{\Delta\omega}{\omega_{\text{пр}}} \ll 1\right)$, then the IFA is usually constructed from identical stages, tuned to the intermediate frequency of $\omega_{\text{пр}}$.

If the number of stages is low, the frequency response of the IFA $H(i\omega)$ can approximate the resonance curve of a single circuit.

In this case.

$$|H(i\omega)|^2 = \frac{K^2}{\frac{\pi^2(\omega - \omega_{\text{пр}})^2}{(\Delta\omega)^2} + 1}, \qquad (2.3.1)$$

where $\Delta\omega$ is the effective pass band of the IFA;

K is the amplification factor of the IFA;

$\omega_{\text{пр}}$ is the intermediate frequency, and $\Delta\omega \ll \omega_{\text{пр}}$.

With a large number of single tuned stages (in practice more than six) the frequency response of the IFA sufficiently well approximates a Gaussian curve

$$|H(i\omega)|^2 = K^2 e^{-\frac{\pi}{\Delta\omega^2}(\omega - \omega_{\text{пр}})^2} \quad (\omega > 0). \qquad (2.3.2)$$

The IFA of broad-band receivers consists of a detuned relatively intermediate frequency of a pair or a set of three stages. In this case, at a given pass

band there is obtained a higher amplification in the stage and a higher selectivity than in the case of resonant IFA'S. With this, the form of the frequency response approaches a square. Therefore, for such IFA'S the frequency response can be considered ideally square:

$$|H(i\omega)|^2 = \begin{cases} K^2 & \text{at} \quad \omega_{\text{пр}} - \frac{\Delta\omega}{2} < \omega < \omega_{\text{пр}} + \frac{\Delta\omega}{2}, \\ 0 & \text{at} \quad \omega < \omega_{\text{пр}} - \frac{\Delta\omega}{2},\ \omega > \omega_{\text{пр}} + \frac{\Delta\omega}{2}. \end{cases} \tag{2.3.3}$$

In virtue of the broad-band characteristics of noise in the input of the receiver, the spectral density of the noise in the output of the IFA is determined as $N_0|H(i\omega)|^2$. Since usually the pass band of the IFA is significantly less than the magnitude of the intermediate frequency, then in the output a narrow-band normal random process will take place. It is not difficult to show that with symmetric frequency response relative to $\omega_{\text{пр}}$ of IFA the function of correlation of the process on the output will have the form

$$R(\tau) = \sigma^2 a(\tau) \cos \omega_{\text{пр}}\tau, \tag{2.3.4}$$

where $a(\tau)$ is the slowly changing function (for the period $\frac{2\pi}{\omega_{\text{пр}}}$) connected with the frequency response of IFA and the spectral density of noise in the input N_0 by the relationship

$$a(\tau) \approx \frac{2}{\sigma^2\pi}\int_0^\infty N_0 |H_1(i\omega)|^2 \cos\omega\tau d\omega, \tag{2.3.5}$$

where $H_1(i\omega) = H(i\omega_{\text{пр}} - i\omega)$;

$\sigma^2 = \frac{1}{\pi}\int_0^\infty |H(i\omega)|^2 N_0 d\omega$ is the dispersion of noise in the output of the IFA.

Producing a calculation for various forms of frequency responses of IFA, we obtain:

—for the resonance curve

$$a(\tau) = e^{-\frac{\Delta\omega}{\pi}|\tau|}; \tag{2.3.6}$$

—for the Gaussian curve

$$a(\tau) = e^{-\frac{\Delta\omega^2}{4\pi}\tau^2}; \tag{2.3.7}$$

—for the ideally square frequency response

$$a(\tau)=\frac{\sin\frac{\tau\Delta\omega}{2}}{\frac{\tau\Delta\omega}{2}}. \qquad (2.3.8)$$

These results will be required henceforth in the determination of the quantitative characteristics of noise on the output of the detector.

Presentation of IFA in the form of a linear quadripole is correct only as long as the amplifier works without overloadings. Since the power of a useful signal and interferences on the input can vary within rather wide limits, for a guarantee of linear operating conditions of IFA and all of the receiving mechanism on the whole there is applied a system of AGC which varies the amplification factor of the receiving mechanism inversely proportional to the amplitude of the input signal. Therefore, IFA can be considered linear within a wide range of variations in amplitude of the input signal.

In spite of the application of a system of AGC, in the IFA there can appear overloadings by strong signals or interferences, with which the IFA can no longer be considered linear. This can occur due to one of following causes:

1. If power of the signal or interference on the input of the system of AGC exceeds the range of power in which it normally works.

2. With large jumps in the level of the signal or interference on the input of the receiver, which can arise, for example, in the mode of survey the AGC system cannot instantly process them and in the first moments of time overloading of the IFA can arise.

3. During the influence of a different form of infrequent pulse interferences of high power, which are not processed by the system AGC.

The phenomenon of overloading the IFA consists of the fact that signals in the input of the last stages of the IFA have such large amplitudes that the characteristics of tubes can no longer be considered linear.

Let us note that in certain receiving mechanisms utilized in the mode of

selection of moving targets, the amplitude limitation in the IFA is introduced intentionally for the purpose of stabilization of the level of a false alarm in the mode of detection of targets in passive interferences.

Considering that in most cases we will be interested in such operating conditions of a receiving mechanism with which the IFA is not overloaded, and also taking into consideration the complexity of the theoretical analysis of the passage of signals and interferences in the mode of overloading through the IFA and all radio channels of the station on the whole, we will henceforth require that the amplifier, at any interference or signal, work in the linear mode and be described by the frequency response of the approximations which were made above.

Let us stop at the problems of gating intermediate frequency amplifiers.

For the purpose of protecting pulse receivers from a different form of non-synchronous pulse interferences gating is applied.

From that which follows it will be clear that gating is connected with the optimum treatment of signals. Gating takes place usually at cutoff of the last stages of the IFA at all times except the time of action of the gates.

As a result of this there takes place a selection of the signal in time. However full cutoff of the IFA stage outside the gate is practically not accomplished owing mainly to direct passage of interferences through the capacitance of the anode the grid of tube, and the assembly capacitances. In a well designed amplifier the attenuation introduced outside the gate by one gating stage of the IFA is obtained on the order of 20 db. Moreover taking into consideration that the crystal mixer is overloaded by strong interferences, three to four gating [illegible] of the IFA appears to be sufficient for selecting interference of practically any power.

In conclusion let us note that in a number of cases during the analysis of influence of a different form of interferences on the radio channel the strobe form will be assumed as square.

2.4. Detection of Noise Concurrently with a Useful Signal

One of the essential nonlinear conversions of noises and a useful signal producible by the receiving mechanism is detection. With this, the characteristics of the signal and noise are changed owing to their interaction. For a subsequent analysis it is necessary to know the mathematical expectation, the spectral density, the correlation function, and also the distributive laws of the probabilities of the random process in the output of the detector. These characteristics are given in the present paragraph.

Depending upon the circuit and operating mode, there can be the following forms of detectors: anode, grid, cathode, and diode. In radar the largest application is received by the diode detector. Considering that all the enumerated types of detectors easily reduce to the diode with certain equivalent parameters we will henceforth consider only the diode detector.

The most important characteristic of the detector is the dependence of the detector current on the applied voltage: $I_д = \varphi(u)$. This dependence is nonlinear. For low input voltages it is well approximated by the function of the form $I_д = \beta u^2$; in this case the detector is quadratic. At high input voltages the dependence of the current of the detector on the applied voltage can be approximated by the function of the form

$$I_д = \varphi(u) = \begin{cases} ku & \text{at} \quad u > 0, \\ 0 & \text{at} \quad u < 0. \end{cases}$$

In this case the detector is called "linear".

During the analysis, the influence of the lag of the load of the detector will be considered approximately assuming that on the output of the detector are only low frequency constituent signals and noises, and high frequency constituents are filtered out in the load and subsequent circuits.

2.4.1. Influence of a Normal Signal of Constant Amplitude Concurrently with Noise on a Linear Detector

Let us analyze the influence of a normal sinusoidal signal of constant amplitude in the presence of set noises of the receiver or broad-band noise interference on a linear detector. Following V. I. Bunimovich [24], the narrow-band random process on the output of the IFA, which takes place owing to the influence of noise, can be represented in the form

$$u(t)=E(t)\cos\Phi(t)=E(t)\cos(\omega_{\text{пр}}t-\theta(t))= \\ =A(t)\cos\omega_{\text{пр}}t+B(t)\sin\omega_{\text{пр}}t, \tag{2.4.1}$$

where $E(t)$ and $\theta(t)$ are the slowly changing random functions of time, which are called accordingly the envelope and phase of the random process;

$A(t)$ and $B(t)$ are the normal random processes with zero mean values and dispersions equal to dispersion σ^2 of the process u(t). In coinciding moments of time, the processes A(t) and B(t) are independent;

$\omega_{\text{пр}}$ is the central frequency of the adjustment of the IFA.

In accordance with that said above, the voltage on the output of the detector will be equal to

$$u_{\text{д}}=\varphi(E\cos\Phi)=\begin{cases} kE\cos\Phi & \text{at} \quad E\cos\Phi>0, \\ 0 & \text{at} \quad E\cos\Phi<0. \end{cases} \tag{2.4.2}$$

Distributing the nonlinear function φ (E cos Φ) in the Fourier series and limiting ourselves to the first member of the series, we find the low-frequency constituent voltage on the output of the detector

$$u_{\text{вых}}(E)=\frac{k}{\pi}\int\limits_{0}^{\pi/2}E\cos\Phi\,d\Phi=\frac{kE}{\pi}. \tag{2.4.3}$$

High frequency constituents do not interest us inasmuch as they will be filtered out by the load.

Consequently, the voltage on the output of the linear detector with accuracy up to a constant factor is equal to the random process envelope on the input. For finding statistical characteristics interesting to us of the random process, on the output it is necessary to know the laws of distribution of the probabilities of the envelope. Let us consider the case where besides the noise on the input of the detector there is also the influence of a sinusoidal signal of constant amplitude E_c, which can be represented in the form

$$u_c(t) = E_c\cos(\omega_{\text{пр}}t - \theta_c) = A_c\cos\omega_{\text{пр}}t + B_c\sin\omega_{\text{пр}}t. \qquad (2.4.4)$$

Then the full expression for the random process on the input of the detector will have the form

$$\begin{aligned} u(t) &= [A(t) + A_c]\cos\omega_{\text{пр}}t + [B(t) + B_c]\sin\omega_{\text{пр}}t = \\ &= E(t)\cos[\omega_{\text{пр}}t - \theta(t)]. \end{aligned} \qquad (2.4.5)$$

Inasmuch as processes $A(t)$ and $B(t)$ in coinciding moments of time are independent, then from the equality (2.4.5) it is simple to see that the problem of finding a one-dimensional distributive law of the envelope $E(t)$ of the sum of the sinusoidal signal and the noise on the input of the detector reduces to finding the probability density of the length of the radius vector, the components of which are independent and have a normal distribution with the parameters $[\sigma, A_c]$ and $[\sigma, B_c]$.

Resolving this problem with the help of the known methods of the theory of probability, it is simple to obtain the expression for a one-dimensional law of distribution of the envelope of the additive mixture of the sinusoidal signal and noise

$$\begin{aligned} f_1(E) &= \frac{E}{\sigma^2} e^{\frac{-(E^2+E_c^2)}{2\sigma^2}} I_0\left[\frac{EE_c}{\sigma^2}\right] && \text{at} \quad E > 0, \\ f_1(E) &\equiv 0 && \text{at} \quad E < 0, \end{aligned} \qquad (2.4.6)$$

where $I_0(x)$ is a Bessel function of the m^{th} order from the imaginary argument.

Thus, the first distribution function of the envelope of the sum of the

sinusoidal signal and noise on the output of the IFA represents the generalized distributive law of Rayleigh.

During the absence of a useful signal $E_c=0$ and the distributive law of the envelope changes to the usual Rayleigh law

$$
\begin{aligned}
f_1(E)&=\frac{E}{\sigma^2}e^{-\frac{E^2}{2\sigma^2}} && \text{at} \quad E>0, \\
f_1(E)&\equiv 0 && \text{at} \quad E<0.
\end{aligned}
\tag{2.4.7}
$$

Henceforth the useful signal will most frequently be in the form of a normal random process. It is clear that the sum of such signal and noise will also be a normal process. Then the distributive law of the envelope of the additive mixture of such a signal and noise will be expressed by the formula (2.4.7), where σ^2 is the dispersion of the sum of the signal and noise.

The two-dimensional distributive law of an envelope in a general case, when there is a normal sinusoidal signal and noise, is described by the following expression:

$$
\begin{aligned}
f_2(E_1, E_2, t, \tau) &= \frac{E_1E_2}{\sigma^4[1-a^2(\tau)]}e^{-\frac{E_1^2+E_2^2+E_{c1}^2+E_{c2}^2-2E_{c1}E_{c2}a(\tau)}{2\sigma^2[1-a^2(\tau)]}}\times \\
&\times\sum_{m=0}^{\infty}\varepsilon_m I_m\left[\frac{a(\tau)E_1E_2}{\sigma^2[1-a^2(\tau)]}\right]I_m\left[\frac{E_{c1}-a(\tau)E_{c2}}{\sigma^2[1-a^2(\tau)]}\cdot E_1\right]\times \\
&\times I_m\left[\frac{E_{c2}-a(\tau)E_{c1}}{\sigma^2[1-a^2(\tau)]}E_2\right],
\end{aligned}
\tag{2.4.8}
$$

where $E_{c1}=E_c(t),\ E_{c2}=E_c(t+\tau)$;
$E_1=E(t);\ E_2=E(t+\tau)$;
$\varepsilon_0=1;\ \varepsilon_m=2$ at $m>0$;

$I_m(x)$ is a Bessel function of the m' order from the imaginary argument.

Inasmuch as the voltage on the output of the detector is proportional to the envelope of the random process on the input, it is obvious that with the help of these expressions the distributive laws of the random process on the output of detector are simply determined.

Using formula (2.4.3) and (2.4.6), for the mathematical expectation of

voltage on the output of the detector we obtain

$$\overline{u_{\text{вых}}(t)} = \frac{k}{\pi}\int_0^\infty E f_1(E)\,dE = \frac{k}{\pi\sigma^2}\int_0^\infty E^2 e^{-\frac{E^2+E_c^2}{2\sigma^2}} I_0\left[\frac{EE_c}{\sigma^2}\right]dE.$$

After integration and necessary conversions we have

$$\overline{u_{\text{вых}}(t)} = \frac{k\sigma}{\sqrt{2\pi}} e^{-\frac{q}{2}}\left\{I_0\left(\frac{q}{2}\right) + q\left[I_0\left(\frac{q}{2}\right) + I_1\left(\frac{q}{2}\right)\right]\right\}, \qquad (2.4.9)$$

where $q = \frac{E_c^2}{2\sigma^2}$ is the relation of signal power and noise on the input of the detector.

The graph of the dependence of the relation $\overline{u_{\text{вых}}(t)}/\overline{u_{\text{вых ш}}(t)}$ on $\sqrt{q}$ is shown in Fig. 2.2. Here the mean value of voltage on the output in absence of a useful signal $|q=0|$ is described by the following expression:

$$\overline{u_{\text{вых ш}}(t)} = \frac{k\sigma}{\sqrt{2\pi}}, \qquad (2.4.10)$$

From the graph it is clear that with high ratios of signal to noise $q > 1$, the curve asymptotically approaches the line $\frac{2}{\sqrt{\pi}}\sqrt{q}$. At small values of $q < 1$ there takes place a suppression of the signal by the noise, since an increase of voltage on the output with weak signals is less than those which were in the absence of noises.

Using the expansions of the Bessel functions we obtain the fact that with small values of $q < 1$ (a weak signal) the relationship (2.4.9) can be written in the form

$$\overline{u_{\text{вых}}(t)} = \frac{k\sigma}{\sqrt{2\pi}}\left(1 + \frac{q}{2} - \frac{q^2}{16} + \dots\right). \qquad (2.4.11)$$

For a strong signal ($q > 1$) the following asymptotic expression will be correct:

$$\overline{u_{\text{вых}}(t)} = \frac{k\sigma}{\pi}\sqrt{2q}\left(1 + \frac{1}{4q} + \frac{1}{32q^2} + \dots\right). \qquad (2.4.12)$$

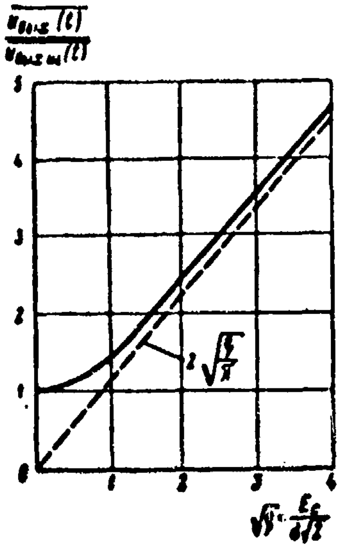

Fig. 2.2. Dependence of the relative mathematical expectation of voltage on the output of the linear detector on the ratio of signal to noise.

Not remaining in detail on the derivation of the formula for the function of correlation of the process of the output of the detector, let us present its final expression

$$R_{\text{вых}}(\tau) \approx \frac{k^2\sigma^2}{8\pi}\left[b_1 a(\tau) + b_2 a^2(\tau)\right], \qquad (2.4.13)$$

where

$$\left.\begin{aligned} b_1 &= 2q\left(1 - \frac{1}{2}q + \frac{3}{16}q^2 - \ldots\right) \\ b_2 &= 1 - q + \frac{11}{16}q^2 - \frac{17}{48}q^3 + \ldots \end{aligned}\right\} \text{ at } q < 1;$$

$$\left.\begin{aligned} b_1 &\to \frac{8}{\pi} \\ b_2 &\to 0 \end{aligned}\right\} \text{ at } q \to \infty.$$

Substituting into formula (2.4.13) $\tau=0$, we obtain the expression for the dispersion of voltage on the output of the detector

$$\sigma^2_{\text{вых}} \approx \frac{k^2\sigma^2}{8\pi}\left[b_1 + b_2\right]. \qquad (2.4.14)$$

Characteristics of the random process in the output of the linear detector in the absence of a sinusoidal signal is simple to obtain by substituting into the preceding formulas q = 0. Then the function of correlation and the dispersion of noise in the output of the detector will have the form

$$R_{\text{вых}}(\tau) \approx \frac{k^2\sigma^2}{8\pi} a^2(\tau), \qquad (2.4.15)$$

$$\sigma^2_{\text{вых}} \approx \frac{k^2\sigma^2}{8\pi}. \qquad (2.4.16)$$

For many practical problems which will be considered in subsequent chapters, it appears to be necessary to know the value of the spectral density of the voltage on the output of the detector at $\omega=0$. Converting, according to Fourier, both parts of the equality (2.4.15), for the spectral density of noise on the output of the linear detector we obtain the following expression:

$$S_{\text{вых}}(\omega) = \frac{k^2\sigma^2}{8\pi}\int_{-\infty}^{\infty} a^2(\tau)\, e^{-i\omega\tau} d\tau.$$

Considering $\omega=0$, we have

$$S_{\text{вых}}(0)=\frac{k^2\sigma^2}{4\pi}\int\limits_0^\infty a^2(\tau)\,d\tau. \qquad (2.4.17)$$

Substituting formulas (2.3.6), (2.3.7) and (2.3.8) into (2.4.17) and effecting integration for various forms of frequency response of the IFA we find:

—for the resonance curve of a single circuit

$$S_{\text{вых}}(0)=\frac{k^2\sigma^2}{8\Delta\omega}; \qquad (2.4.18)$$

—for the Gaussian curve

$$S_{\text{вых}}(0)=\frac{k^2\sigma^2}{4\sqrt{2}\Delta\omega}; \qquad (2.4.19)$$

—for the square frequency response

$$S_{\text{вых}}(0)=\frac{k^2\sigma^2}{4\Delta\omega}. \qquad (2.4.20)$$

2.4.2. Characteristics of Noise on the Output of a Square-Law Detector

Let us consider the problem of the influence of noise on a square-law detector. Noise on the input of the detector, as before will be considered a stationary, narrow-band, normal random process with a zero mean value and dispersion equal to σ^2 .

The noise voltage after nonlinear conversion in the detector (without calculation of filtration in the load) $u_{\text{д}}(t)$ will be connected with the voltage on the input of the detector by the relationship

$$u_{\text{д}}(t)=ku^2(t). \qquad (2.4.21)$$

Than it is obvious that the mathematical expectation of noise on the output of the square-law detector will have the form

$$\overline{u_{\text{вых}}(t)}=\overline{u_{\text{д}}(t)}=k\sigma^2. \qquad (2.4.22)$$

Using formulas (2.4.21) and (2.4.22), the function of correlation of the

voltage $u_{\text{д}}(t)$ can be written in the form

$$R_{\text{д}}(\tau)=\overline{u_{\text{д}}(t)\,u_{\text{д}}(t+\tau)}-\overline{u_{\text{д}}(t)}^{2}=$$
$$=k^{2}[\overline{u^{2}(t)\,u^{2}(t+\tau)}-\sigma^{4}].$$

Let us use the formula for the compound central moment of the fourth order of normally distributed random variables

$$\overline{x_1x_2x_3x_4}=R_{12}R_{34}+R_{13}R_{24}+R_{14}R_{23}, \tag{2.4.23}$$

where R_{ij} is the correlated moment of random variables x_i and x_j. Designating

$$x_1=x_2=u(t),\quad x_3=x_4=u(t+\tau)$$

and using the obtained relationships, the expression for the function of correlation of voltage $u_{\text{д}}(t)$ can be written in the form

$$R_{\text{д}}(\tau)=2k^{2}R^{2}(\tau). \tag{2.4.24}$$

Substituting expression (2.3.4) for $R(\tau)$ into (2.4.24) and considering that the filter of the detector will separate only low-frequency constituents of the voltages of noises, we obtain the function of correlation of the voltage on the output of the detector in the form

$$R_{\text{вых}}(\tau)=k^{2}\sigma^{4}a^{2}(\tau). \tag{2.4.25}$$

Hence the dispersion of noise on the output is equal to

$$\sigma^{2}_{\text{вых}}=k^{2}\sigma^{4}. \tag{2.4.26}$$

As was already noted, for an analysis of the influence of set noises of the receiving mechanism or noise interference on the radio channel of the radar set, it is necessary to calculate the value of the spectral density of noise on the output of the detector at $\omega=0$. Converting, according to Fourier, both parts of the equality (2.4.25), for the spectral density of noise on the output we find

$$S_{\text{вых}}(\omega)=k^{2}\sigma^{4}\int_{-\infty}^{\infty}a^{2}(\tau)\,e^{-j\omega\tau}d\tau. \tag{2.4.27}$$

Considering $\omega = 0$, we obtain

$$S_{\text{вых}}(0) = 2k^2\sigma^4 \int_0^\infty a^2(\tau)\, d\tau. \tag{2.4.28}$$

Calculating $S_{\text{вых}}(0)$ analogously to the preceding, and for various forms of frequency response of IFA, we obtain:

—for the resonance curve of a single circuit

$$S_{\text{вых}}(0) = 2k^2\sigma^4 \frac{\pi}{2\Delta\omega}; \tag{2.4.29}$$

—for the Gaussian curve

$$S_{\text{вых}}(0) = 2k^2\sigma^4 \frac{\pi}{\Delta\omega\sqrt{2}}; \tag{2.4.30}$$

—for a square frequency response

$$S_{\text{вых}}(0) = 2k^2\sigma^4 \frac{\pi}{\Delta\omega}. \tag{2.4.31}$$

2.4.3. The Influence of Noise Concurrent, with a Signal on a Square-Law Detector

Let us now consider the detection of noise in the presence of a signal. Let the input of the detector be influenced by the sum of the signal s(t) and noise n(t)

$$u(t) = s(t) + n(t). \tag{2.4.32}$$

The signal will not be specified for now and the noise, as before, will be considered a narrow-band, normal random process with a mean value of zero and a dispersion of σ^2. Naturally, the signal and noise are independent.

The voltage on the output of the detector will be recorded in the form

$$u_{\text{д}}(t) = ku^2(t) = k[s(t) + n(t)]^2. \tag{2.4.33}$$

The mathematical expectation of this voltage is equal to

$$\overline{u_{\text{д}}(t)} = k[\overline{s^2(t)} + \overline{n^2(t)}]. \tag{2.4.34}$$

Using (2.4.33) and (2.4.34), it is simple to find the function of correlation

of the random process on the output of the square-law detector

$$R_{\text{д}}(t,\tau)=\overline{u_{\text{д}}(t)\,u_{\text{д}}(t+\tau)}-\overline{u_{\text{д}}(t)}\;\overline{u_{\text{д}}(t+\tau)}=$$
$$=k^2\left[R_{s^2}(\tau)+R_{n^2}(\tau)+4R_n(\tau)\,\overline{s(t)\,s(t+\tau)}\right], \qquad (2.4.35)$$

where the component

$$R_{s^2}(\tau)=\overline{s^2(t)\,s^2(t+\tau)}-\overline{s^2(t)}\;\overline{s^2(t+\tau)}$$

resulted from the presence of the signal; the second component

$$R_{n^2}(\tau)=\overline{n^2(t)\,n^2(t+\tau)}-\overline{n^2(t)}\;\overline{n^2(t+\tau)}$$

resulted from the presence of noise, and the third component appeared owing to the interaction of the signal with the noise. In the derivation of this expression it was considered that $\overline{n(t)}=\overline{n(t+\tau)}=\overline{n(t)\,n^2(t+\tau)}=\overline{n^2(t)\,n(t+\tau)}=0$ were like the central moments of an odd order of normally distributed random variables.

Let us now consider the specific forms of a useful signal. In Chapter 1 it was shown that the signal reflected from a target can be represented in most cases in the form of a narrow-band normal random process. If the function of correlation of a signal on the output of the IFA is equal to $\sigma_c^2 b(\tau)\cos\omega_{\text{пр}}\tau$, then from the formula (2.4.34) it follows that the mathematical expectation of the random process on the output of the detector will be equal to

$$\overline{u_{\text{д}}(t)}=k\left[\sigma_c^2+\sigma^2\right]. \qquad (2.4.36)$$

Thus, an increase of the constant constituent of voltage on the output of the detector is proportional to the dispersion of the useful signal on the input.

Let us turn to a determination of the function of correlation of the voltage on the output of the detector.

Since the sum of the signal and noise in the given case represents a normal random process, then for the function of correlation $R_{\text{вых}}(\tau)$ formula (2.4.25) is correct in which $\sigma^4 a^2(\tau)$ must be replaced by $\left[\sigma_c^2\, b(\tau)+\sigma^2 a(\tau)\right]^2$, as a result

of

$$R_{\text{вых}}(\tau)=k^2[\sigma_s^2 b(\tau)+\sigma^2 a(\tau)]^2. \qquad (2.4.37)$$

Consequently, the function of correlation of the voltage on the output of a square-law detector is proportional to the square of the sum of the envelope functions of correlations of a useful signal and noise on its input.

Dispersion of the process on the output of the detector is

$$\sigma^2_{\text{вых}}=k^2[\sigma_s^2+\sigma^2]^2. \qquad (2.4.38)$$

Comparing formula (2.4.38) and (2.4.36), it is simple to note that the dispersion of the random process on the output of a square-law detector in the case considered is equal to the square of its mathematical expectation.

In the case where the useful signal is a sinusoidal oscillation

$$s(t)=E_c\cos\omega_{\text{пр}}t,$$

the mathematical expectation $\overline{u_{\text{д}}(t)}$ can be found, again using the general formula (2.4.34),

$$\overline{u_{\text{д}}(t)}=k\left[\frac{E_c^2}{2}+\sigma^2+\frac{E_c^2}{2}\cos 2\omega_{\text{пр}}t\right].$$

Since the high-frequency constituents will be filtered out, the final expression for the mathematical expectation of the voltage on the output of the detector will have the form

$$\overline{u_{\text{вых}}(t)}=k\sigma^2[1+q]. \qquad (2.4.39)$$

where q is the above introduced relation of the power of the signal and noise on the input of the detector.

The function of correlation of the process on the output of the detector according to the formula (2.4.35). It is obvious that at $s(t)=E_c\cos\omega_{\text{пр}}t$

$$\overline{s(t)s(t+\tau)}=E_c^2\cos\omega_{\text{пр}}t\cos[\omega_{\text{пр}}(t+\tau)],$$
$$R_{s\text{ш}}(t,\tau)=0.$$

The remaining constituents of formula (2.4.35) we have already calculated. Then

$$R_{\text{д}}(t,\tau)=2k^2[R_{\text{ш}}^2(\tau)+2E_c^2\cos\omega_{\text{пр}}t\cos[\omega_{\text{пр}}(t+\tau)]R_{\text{ш}}(\tau)].$$

Since the filter standing in the load of the detector neutralizes the random process in time, we will be interested in the function of correlation $\widetilde{R_2(t,\tau)}$ neutralized in time.

Considering that the average in time is

$$\widetilde{\cos\omega_{\text{пр}}(t)\cos[\omega_{\text{пр}}(t+\tau)]}=\frac{1}{2}\cos\omega_{\text{пр}}\tau,$$

we obtain

$$\widetilde{R_2(t,\tau)}=2k^2[R_{\text{н}}^2(\tau)+E_{\text{c}}^2R_{\text{н}}(\tau)\cos\omega_{\text{пр}}\tau]. \qquad (2.4.40)$$

Substituting into (2.4.40) the expression for the function of correlation of noise (2.3.4) and being interested only in the low-frequency constituents of voltage on the output, we find the final expression for the function of correlation of the voltage on the output of a square-law detector

$$R_{\text{вых}}(\tau)=k^2\sigma^4[a^2(\tau)+2qa(\tau)]. \qquad (2.4.41)$$

The dispersion of low-frequency constituents of the voltage on the output of a detector is

$$\sigma_{\text{вых}}^2=k^2\sigma^4[1+2q]. \qquad (2.4.42)$$

It is useful to note that the intensity of low-frequency constituents on the output of a square-law detector consists of two parts. The first part does not depend on the useful signal and is caused only by noises. The second part is caused by the interaction between noise and a useful signal at detection.

Let us find the distributive laws of the voltage on the output of a detector. Applying the envelope method, it is easy to show that the low-frequency part of the voltage on the output of a square-law detector is proportional to the square of the envelope E(t) of the random process on the input. The probability densities of the envelope for cases interesting us will be expressed by formulas (2.4.6), (2.4.7), (2.4.8). Consequently, it is necessary to find the distributive laws of the random process x(t), which is connected with E(t) by the relationship

$$x(t)=E^2(t).$$

if the distributive laws of the process E(t) are known.

Applying the rules of the theory of probability we obtain the expression for a two-dimensional probability density of the square of the envelope of the random process consisting of the additive mixture of the sinusoidal signal and noise [17]:

$$f_2(x_1, x_2, t, \tau) = \frac{1}{4\sigma^4[1-a^2(\tau)]} \times$$
$$\times e^{-\frac{x_1+x_2+E_{c1}^2+E_{c2}^2-2E_{c1}E_{c2}a(\tau)}{2\sigma^2[1-a^2(\tau)]}} \sum_{m=0}^{\infty} \varepsilon_m I_m\left[\frac{a(\tau)\sqrt{x_1x_2}}{\sigma^2[1-a^2(\tau)]}\right] \times$$
$$\times I_m\left[\frac{E_{c1}-E_{c2}a(\tau)}{\sigma^2[1-a^2(\tau)]}\sqrt{x_1}\right] I_m\left[\frac{E_{c2}-E_{c1}a(\tau)}{\sigma^2[1-a^2(\tau)]}\sqrt{x_2}\right]$$
$$\text{at } x_1 > 0 \text{ and } x_2 > 0, \qquad (2.4.43)$$

where $x_1 = E^2(t)$; $x_2 = E^2(t+\tau)$; the remaining designations are the same as in the formula (2.4.8).

In the case where a useful signal is absent, the two-dimensional distributive law of the random process on the output of a square-law detector will be determined from (2.4.43), where it is necessary to place $E_{c1} = E_{c2} = 0$,

$$f_2(x_1, x_2, \tau) = \frac{1}{4\sigma^4[1-a^2(\tau)]} e^{-\frac{x_1+x_2}{2\sigma^2[1-a^2(\tau)]}} \times$$
$$\times I_0\left[\frac{a(\tau)\sqrt{x_1x_2}}{\sigma^2[1-a^2(\tau)]}\right]$$
$$\text{where } x_1 > 0 \text{ and } x_2 > 0. \qquad (2.4.44)$$

The one-dimensional probability density of the square of the envelope of the sum of the sinusoidal signal and noise can be obtained from (2.4.43). It has the form

$$f_1(x, t) = \frac{1}{2\sigma^2} e^{-\frac{x+E_c^2}{2\sigma^2}} I_0\left(\frac{E_c\sqrt{x}}{\sigma^2}\right) \text{ where } x > 0. \qquad (2.4.45)$$

For the case where a useful signal is absent, from formula (2.4.45) at $E_c = 0$ we will obtain a one-dimensional probability density of the square of the noise envelope.

Let us note that the distributive laws of probabilities of square

of the envelope of the sum of the useful signal in the form of a normal random process and noise will also be described by formulas (2.4.44) and (2.4.45), where the parameters σ^2 and $a(\tau)$ refer to the sum of the signal and noise.

Thus, we have considered the main statistical characteristics of the random processes on the output of the detector. These characteristics will be required in subsequent chapters.

2.5. On the Passage of a Signal and Interferences Through a Video Amplifier and a Low Frequency Amplifier

After detection, the signal proceeds to the input of the video amplifier in radar sets with incoherent reception or a low frequency amplifier in coherent receivers. These amplifiers can be of the most divers forms.

Not remaining on their specific peculiarities, let us consider the idealizations which will be used henceforth, and the question about to what measure these idealizations are carried out in practice.

First of all let us note that the amplitude characteristics of these amplifiers can be considered linear only within the limited range of changes of the input signal. An analysis of the radio channel will be conducted in subsequent chapters on the assumption that the amplitude characteristic of amplifiers is linear up to a certain value of amplitude of signal in the input, and then there begins the ideal limitation (Fig. 2.3).

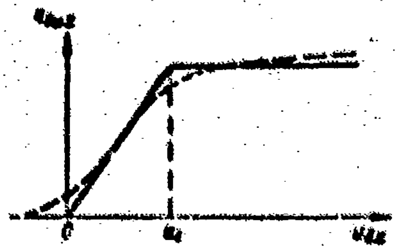

Fig. 2.3. Amplitude characteristic of an amplifier.

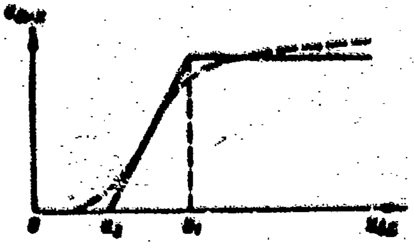

Fig. 2.4. Amplitude characteristic of an amplifier.

In certain amplifiers of a special form, along with the limitation above, a certain threshold cutoff or limitation is introduced from below. The exemplary form of the amplitude characteristic of such an amplifier, the broken-line curve approximating it is shown in Fig. 2.4.

Let us note that since usually the amplification factor of a video amplifier or low frequency amplifier becomes rather high, the level of the signal limitation in the receiving mechanism on the whole, in most cases, is determined by the limitation in these amplifiers.

The signal from the output of a video amplifier or AFA usually proceeds to the input of circuits, the inertness of which is significantly higher than the inertness of an amplifier. For example, in channels of tracking by angular coordinates, in the stations using the method of conical scanning or method of tracking by bundles of pulses, the signal moves to the input of the pulse detector The pass band of the pulse detector on the envelope is significantly less than the pass band of a video amplifier.

In the channels of range tracking in time, the discriminator is a significant more inertial mechanism than the video amplifier, etc. Therefore, in an analysis of the influence of interferences on the radio channel the video amplifier or AFA will be considered an inertialess link with a broken-line amplitude curve.

Since the signal on the input of these amplifiers is much larger than in an intermediate frequency amplifier then they, to large degrees, are subject to overloadings. The phenomenon of overloading consists basically of the fact that during the income of strong signals or interferences, the transient and coupling capacitors are charged. Due to this, after the income of such interference there occurs a temporary decrease of amplification. If for the duration of this time a useful signal appears, then it will not practically pass through the amplifier.

However, in most cases we will be interested in such operating modes of the video amplifier or AFA when they are not overloaded and, considering that in

their construction special circuits are applied for controlling the consequences of overloadings, we will henceforth not take these phenomena into account.

2.6. Characteristics of Voltage on the Output of Phase and Pulse Detectors

Important and very frequently utilized elements of radio channels are phase detectors. They are applied in various goniometer and frequency meter systems an analysis of which will be conducted in subsequent chapters. Usually phase detectors are divided into two groups: switching and vector measuring.

The diagram of a switching phase detector is in Fig. 2.5. Its characteristic peculiarity is the presence of a high-speed electronic commutator (of the forming stage), which with the period of the reference voltage $u_2(t)$ given it switches and disconnects the amplifying tubes. The operation of the commutator is completely unconnected with the input voltage $u_1(t)$. Consequently, the circuit appears to be linear on the input voltage, and the voltage to the low-frequency filter $u_3(t)$ is expressed by the formula

$$u_3(t)=u_1(t)v(t),$$

where v(t) is the square voltage obtained from the reference limitation.

If the reference voltage is not subject to interferences, and in the spectrum $u_1(t)$ the components on the third harmonic of the pedestal frequencies are small, then, as it was shown in [30] $u_3(t)$ it is possible to consider it originating from $u_1(t)$ by means of multiplication by the main harmonic of the reference voltage. Commutational phase detectors usually are used at low frequencies (in circuits of goniometric channels of stations with conical scanning).

At high frequencies, for example in circuits of instantaneous comparison of signals, more frequently used are vector-measuring detectors, a diagram of which is shown in Fig. 2.6. On diode D_1 is the sum of the voltages $u_1(t)$ and $u_2(t)$, and on diode D_2 is the difference of the same voltages. The voltages received as a result of detection are subtracted by means of special switching of the

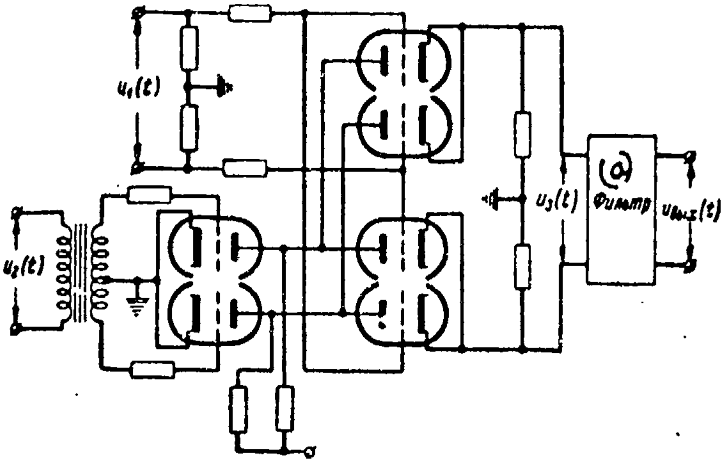

Fig. 2.5. Diagram of a phase detector of the switching type.
KEY: (a) Filter.

loads of the amplitude detectors.

The presence of diodes in combination with RC-nets makes a circuit, strictly speaking nonlinear relative to $u_1(t)$. However, considering the amplitude detectors as mechanisms separating the envelope or square of the envelope of the input narrow-band random process, it is easy to obtain the mathematical operations carrying out the phase detector of such a type.

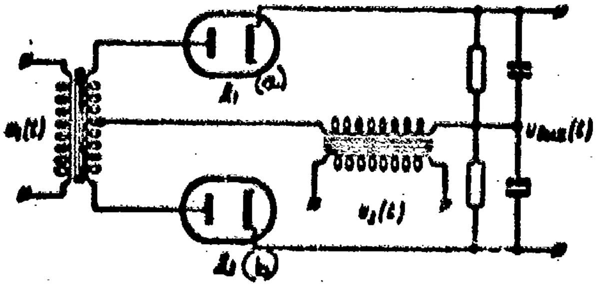

Fig. 2.6. Diagram of a phase detector of the vector measuring type.
KEY: (a) Diode 1; (b) Diode 2.

With quadratic amplitude detectors, designating

$$u_1(t) = E_1(t)\cos[\omega_{пр}t + \varphi_1(t)],$$
$$u_2(t) = E_2(t)\cos[\omega_{пр}t + \varphi_2(t)],$$

for the square of the envelope of the sum and the difference of these voltages we obtain

$$E_{+}^{2}(t)=E_{1}^{2}(t)+2E_{1}(t)E_{2}(t)\cos[\varphi_{1}(t)-\varphi_{2}(t)]+E_{2}^{2}(t),$$
$$E_{-}^{2}(t)=E_{1}^{2}(t)-2E_{1}(t)E_{2}(t)\cos[\varphi_{1}(t)-\varphi_{2}(t)]+E_{2}^{2}(t),$$

where $E_{+}(t)$ is the envelope of $u_2(t)+u_1(t)$;

$E_{-}(t)$ is the envelope of $u_2(t)-u_1(t)$.

Hence, considering the presence of subtraction in the circuit, for voltage on the output of the phase detector we have the following expression:

$$u_{вых}(t)=k_{2}E_{1}(t)E_{2}(t)\cos[\varphi_{1}(t)-\varphi_{2}(t)]. \qquad (2.6.1)$$

where k_2 is the proportionality factor, having the dimension $\left[\frac{1}{в}\right]$.

Thus a phase detector with quadratic characteristics of diodes is equivalent to a simple cross-multiplying mechanism (with further rejecting of higher harmonics of $\omega_{оп}$). With linear characteristics of diodes the equivalences of the multiplication of input voltages is not obtained. In this case, as is easy to show, the phase detector is equivalent to the cross-multiplied voltage $u_1(t)$ with the standardized reference voltage (again with rejection of the higher harmonics $\omega_{оп}$).

Let us define the characteristics of the random process of a phase detector with quadratic characteristics of the diodes. Let on one input be the sum of the signal s(t) and noise $n_1(t)$

$$u_{1}(t)=s(t)+n_{1}(t).$$

On the other input

$$u_{2}(t)=ms(t)+n_{2}(t).$$

where $n_2(t)$ is the noise voltage, uncorrelated with $n_1(t)$;

m is the positive value, smaller than one.

Such a position takes place, for example, in radar sets with coherent radiation using the method of instantaneous amplitude comparison of signals (see Ch. 10). The useful signal s(t) represents a narrow-band normal random process.

Then from formula (2.4.23) it follows that the function of the correlation of process obtained as a result of multiplication of $u_1(t)$ and by $u_2(t)$ will equal to

$$R(\tau)=R_{11}(\tau)R_{22}(\tau)+R_{21}(\tau)R_{12}(\tau),$$

where $R_{11}(\tau)$ and $R_{22}(\tau)$ is the function of correlation of random processes on two inputs accordingly;

$R_{21}(\tau)$ and $R_{12}(\tau)$ is the function of mutual correlation.

In view of narrow-band characteristic of processes these functions of correlation will have the form of (2.3.4).

Considering that the high-frequency constituents of the process in the output will be filtered out by the detector load and subsequent inertial circuits, for the function of correlation of voltage on the output of the detector we obtain the following expression:

$$R_{\text{вых}}(\tau)=\frac{1}{2}[\sigma_1^2\sigma_2^2 a_{11}(\tau)a_{22}(\tau)+\sigma_1^2\sigma_2^2 a_{21}(\tau)a_{12}(\tau)], \qquad (2.6.2)$$

where analogous to expression (2.3.5)

$$\sigma_1^2 a_{11}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}2|H_1(i\omega)|^2[m^2S_c(\omega)+N_0]\cos\omega\tau d\omega;$$

$$\sigma_2^2 a_{22}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}2|H_1(i\omega)|^2[S_c(\omega)+N_0]\cos\omega\tau d\omega;$$

$$\sigma_1\sigma_2 a_{12}(\tau)=\sigma_1\sigma_2 a_{21}(\tau)=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}2|H_1(i\omega)|^2 S_c(\omega)\cos\omega\tau\, d\omega. \qquad (2.6.3)$$

Here $H_1(i\omega)$ and $S_c(\omega)$ are low-frequency equivalents of the frequency response of IFA and the spectral density of the useful signal on the input of IFA; N_0 is as before, the spectral density of the input noise.

It is obvious that the mathematical expectation of the voltage on the output

of the phase detector will have the form

$$\overline{u_{\text{вых}}(t)} = R_{12}(0) = \sigma_1\sigma_2 a_{12}(0) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} 2\,|H_1(i\omega)|^2 S_1(\omega)\,d\omega. \qquad (2.6.4)$$

Since subsequent circuits are very inertial in relation to the correlation time of the process on the output, then a sufficient characteristic of the process will be the value of the spectral density at zero frequency $S_{\text{вых}}(0)$. Analogous to the preceding

$$S_{\text{вых}}(0) = \frac{\sigma_1^2\sigma_2^2}{2\pi}\int\limits_{-\infty}^{\infty}\frac{1}{2}\left[a_{11}(\tau)a_{22}(\tau) + a_{12}(\tau)a_{21}(\tau)\right]d\tau.$$

Using equality (2.6.3), according to the Parseval theorem we obtain

$$S_{\text{вых}}(0) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}|H_1(i\omega)|^4\left\{\left[m^2S_1(\omega) + N_0\right]\left[S_1(\omega) + N_0\right] + \right.$$
$$\left. + m^2S_1^2(\omega)\right\}d\omega. \qquad (2.6.5)$$

On the output of the receiving mechanism in the angle measuring channels of the stations using the method of conical scanning or the method of tracking by bundles of pulses, is placed the pulse detector which should separate the envelope of pulses. The circuit of the detector is shown in Fig. 2.7. Usually the time constant of the charge, determined by the product of $(R_0+R_i)C_\partial$, is much longer than the duration of the gate. Therefore, it is possible to consider that the detector integrates the input voltage during the time of gate action.

Consequently the voltage on the output of the detector to the end of the action of the strobe is equal to

$$u_{\text{вых}} = \frac{k}{\tau_{\text{и}}}\int\limits_0^{\tau_{\text{и}}} u_{\text{вх}}(t)\,dt. \qquad (2.6.6)$$

As was shown in [31] the pulse detector, during the absence of overloadings, is equivalent by the envelope to a simple RC-filter with an equivalent time constant determined by the formula

$$T_{\text{э}} = \frac{TT_1}{T_1 + \gamma T}, \qquad (2.6.7)$$

where T is the time constant of discharge;

T_1 is the time constant of charge;

γ is the space charge coefficient.

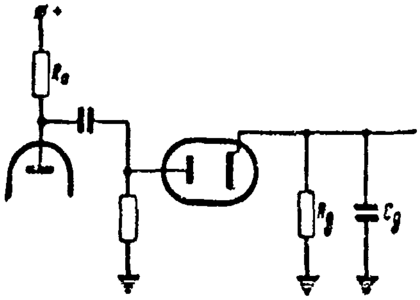

Fig. 2.7. Circuit of a pulse detector.

In connection with this, the pulse detector will henceforth be frequently considered equivalent to a circuit consisting of a series connected amplifier with an amplification factor equal to the transmission factor of the detector and an RC-filter with a time constant $T_{э}$. In view of the large inertness of subsequent circuits, the inertness of the detector can frequently be disregarded.

More accurate analysis of the work of the pulse detector, taking into account the presence of overloadings, is contained in [32].

The information shown on phase and pulse detectors is sufficient for an analysis of a radio channel by the influence of various forms of signals and interferences.

2.7. Influence of Signals and Interferences on a Receiving Mechanism with Automatic Gain Control

The system of automatic gain control (AGC) is a part of the majority of receiving mechanisms of radar stations. From practice it is known that the AGC system significantly affects the passage of signals and interferences through the radio receiving mechanism.

Depending upon the selection of the parameters of the AGC, the fluctuation of the signal can be smoothed or increased owing to its operation. Physically, this occurs because under the influence of an input signal on the output of the filter of the AGC the adjustment voltage is produced which contains to some degree the fluctuating constituent. This voltage changes the amplification factor of

the receiving mechanism as a result of which the random constituents of the signal on the output of the corresponding radar channel can be decreased as well as increased. On the operation of the AGC system depends the accuracy of the radar measurements in the presence of interferences.

Theoretically, the problem of investigating the influence of random voltages on a receiving mechanism with AGC appears to be rather complicated inasmuch as the AGC system is nonlinear. Even with linearization of the regulating characteristic of the receiver system the AGC is described by differential equations with variable coefficients depending on the input signal. With the influence of the random process on the input of a receiving mechanism with AGC, its parameters change in a random manner and they are correlated with the input influence. Therefore, a strict resolution of the problem within the frame work of the correlation theory is impossible.

In the present paragraph the influence will be thoroughly analyzed on the AGC system of noises and fluctuations of a signal reflected from a target, pulse random interference, intermittent interferences, and also a signal fluctuating in amplitude and modulated by sinusoidal law (which takes place in radar sets using the method of conical scanning). As a result of the analysis there the characteristics are obtained of the fluctuations and interferences on the output of the receiving mechanism.

Let us note that in subsequent chapters the properties of AGC will be considered simplified since a strict calculation of the properties of the AGC system leads to a significant complication of the analysis of the radio channels of the measuring systems of coordinates. With this a manifestation of the main regularities is hampered.

However during the analysis of the radio channels of specific radar sets with a quantitative appraisal of the noise proof feature, a simplified approach to the appraisal of the influence of AGC frequently turns out to be insufficient.

More accurate expressions are required for the characteristics of the signal and the interferences in the output of the receiver. In these cases the results of the conducted analysis will appear useful and necessary.

2.7.1. Influence of Signal Fluctuations on a Receiving Mechanism with AGC

An equivalent circuit of the AGC system with delay is shown in Fig. 2.8. The voltage on the input u(t) moves to a variable amplifier. If the voltage on the output of the amplifier v(t) exceeds the level of delay E_3, then on the output of the feedback circuit is produced an adjustment voltage $E_p(t)$, which changes the amplification factor of the receiving mechanism in such away in order to ensure small changes in the magnitude of the voltage on the output with large changes of voltage on the input.

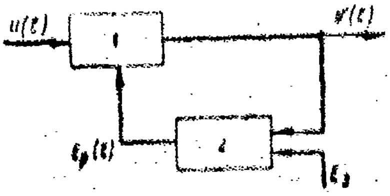

Fig. 2.8. Equivalent circuit of a receiver with AGC:
1) variable amplifier; 2) feedback circuit.

During the analysis it will be assumed that the variable amplifier is inertialless since the inertness of the feedback circuit is much higher than the inertness of the amplifier. In virtue of the high inertness, the input perturbation of the AGC system may be considered the envelope of the signal on the input even in a case where the signal has a pulse character. In accordance with this, by u(t) and v(t) everywhere will henceforth be understood as the envelopes of these signals. The feedback circuit is assumed to be linear for the envelope of the signal. This means that the detector in the feedback circuits is either inertialless or is equivalent to a linear inertial circuit with respect to the envelope [31]. ~~Furthermore~~, it is assumed that owing to the fluctuations the amplitude of the signal does not fall lower than the level of delay. This assumption is fully permissible if the delay is carried out after the filter in the feedback circuits of the AGC.

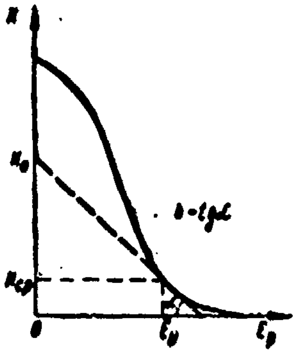

Fig. 2.9. Regulating characteristic of an amplifier.

In the solution of the problem we will use a piecewise linear approximation of the regulating characteristic by which we will understand the dependence of the amplification factor on the voltage of adjustment. Then on every section of approximation the amplification factor will be expressed by the formula

$$K(E_p) = K_0 - bE_p, \qquad (2.7.1)$$

where K_0 and $b = \operatorname{tg} \alpha$ are the parameters of the linearized regulating characteristic on a given section of the approximation, the value of which is clear from Fig. 2.9.

Since the steady-state mode of operation of the system interests us it is sufficient to consider one section of approximation corresponding to the given established state of the system. Thus, the voltage on the output of the AGC system is connected with the voltage on the input by the relationship

$$v(t) = u(t)[K_0 - bE_p(t)]. \qquad (2.7.2)$$

The feedback circuit, in virtue of its linearity, is described by the frequency response $k_1 H(i\omega)$, where $H(0) = 1$.

The random perturbation on the input is conveniently written in the form

$$u(t) = A + \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega) e^{i\omega t} d\omega, \qquad (2.7.3)$$

where $A = \overline{u(t)}$ is the mathematical expectation of the random process on the input; $U(\omega)$ is the random function ω.

The random process on the output of the system is conveniently presented in the form

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\omega) e^{i\omega t} d\omega. \qquad (2.7.4)$$

The adjustment voltage, in view of the linearity of the feedback circuit with respect to the envelope, will be determined by the formula

$$E_p(t)=\frac{k_1}{2\pi}\int\limits_{-\infty}^{\infty}V(\omega)H(i\omega)e^{i\omega t}d\omega-k_1E_0. \tag{2.7.5}$$

From formulas (2.7.2)—(2.7.5) it is easy to obtain an integral equation for the complex spectrum of the random process on the output of the receiving mechanism with AGC

$$V(\omega)=\frac{K_0+bk_1E_0}{1+Abk_1H(i\omega)}\left[2\pi A\delta(\omega)+U(\omega)\right]-$$
$$-\frac{bk_1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{V(x)H(ix)U(\omega-x)\,dx}{1+Abk_1H(i\omega)}, \tag{2.7.6}$$

where $\delta(\omega)$ is the delta-function.

The obtained equation completely describes the processes in the AGC system of any order in the steady-state operating mode of operation.

An accurate resolution of this integral equation for an arbitrary form of the nucleus is unknown; therefore, we seek the resolution in the form of successive approximations

$$V(\omega)=V_0(\omega)+V_1(\omega)+V_2(\omega)+V_3(\omega)+\ldots \tag{2.7.7}$$

in an assumption of the smallness of the dispersion of fluctuations as compared with the square of the mean value of the signal on the input $\left(\frac{\sigma^2}{A^2}<1\right)$. Then the zero approximation will be written in the form

$$V_0(\omega)=\frac{K_0+bk_1E_0}{1+Abk_1H(i\omega)}2\pi A\delta(\omega). \tag{2.7.8}$$

and the first correction to it

$$V_1(\omega)=\frac{(K_0+bk_1E_0)U(\omega)}{1+Abk_1H(i\omega)}-$$
$$-\frac{bk_1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{V_0(x)H(ix)U(\omega-x)\,dx}{1+Abk_1H(i\omega)}.$$

Substituting into this formula the expression for $V_0(\omega)$, after simple

conversions we obtain

$$V_1(\omega)=U(\omega)H_э(i\omega), \tag{2.7.9}$$

where

$$H_э(i\omega)=\frac{K_{cp}}{1+Abk_1H(i\omega)}, \tag{2.7.10}$$

$$K_{cp}=\frac{K_0+bk_1\Xi_3}{1+Abk_1} \tag{2.7.11}$$

-- is the mean value of the amplification factor of the variable amplifier.

Analogously, we will find the expressions for the second and third correction:

$$V_2(\omega)=-\frac{b}{2\pi K_{cp}}\int\limits_{-\infty}^{\infty}U(s)U(\omega-s)H_p(is)H_э(i\omega)ds, \tag{2.7.12}$$

$$V_3(\omega)=\left(\frac{b}{2\pi K_{cp}}\right)^2\iint\limits_{-\infty}^{\infty}U(\omega-s)U(s')U(s-s')H_p(is)\times$$

$$\times H_p(is')H_э(i\omega)dsds', \tag{2.7.13}$$

where

$$H_p(i\omega)=k_1H(i\omega)H_э(i\omega). \tag{2.7.14}$$

The obtained solution gives the possibility to determine the statistical characteristics of the random process in the output of a receiver with ACC. Indeed, the mathematical expectation of the signal on the output can be found from the relationship

$$\overline{v(t)}=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\overline{V(\omega)}e^{i\omega t}d\omega. \tag{2.7.15}$$

Substituting the solution for V(ω), we obtain the mathematical expectation of the random process on the output of a receiver with accuracy up to the second approximation in the form

$$\overline{v(t)}\approx K_{cp}A-\frac{b}{2\pi(1+Abk_1)}\int\limits_{-\infty}^{\infty}S_u(\omega)H_p(i\omega)d\omega. \tag{2.7.16}$$

where $S_u(\omega)$ is the spectral density of the fluctuations on the input of a receiver

In the conclusion of this formula the known relationship was used for the stationary random process

$$\overline{U(x)U(\omega-x)}=2\pi S_u(x)\,\delta(\omega).$$

Since the magnitude of the product Abk_1 is usually much higher than one, then with a sufficient degree of accuracy for pratice it is possible to consider that the mathematical expectation of the fluctuating signal on the output of the receiving mechanism will be equal to

$$\overline{v(t)}\approx K_{cp}A. \tag{2.7.17}$$

Thus, if the obtained idealizations are fulfilled then the mean value of the fluctuating signal on the output of a receiving mechanism with AGC will be approximately equal to the mathematical expectation of the signal on the input multiplied by the average amplification factor of the receiving mechanism.

Using the expression for K_{cp}, the formula (2.7.17) can be approximately recorded in the form

$$v(t)\approx K_{cp}A\approx E_3+\frac{K_\phi}{bk_1}\approx \text{const}.$$

Since the magnitude $\frac{K_\phi}{bk_1}$, representing the static error of AGC is usually much less than E_3, then the mean value of the random process on the output can be considered approximately constant and equal to the product of the amplification factor of the receiver by the mathematical expectation of the random process on the input in that range of power of the input signal or interferences in which the AGC system normally operates. If the random process on the input represents a mixture of the useful signal and the interference, then from that presented it ensues that with an increase in the power of the interference the part of the useful signal on the output will decrease. As will be clear from the following in the treatment of the radar tracking meters (see Ch. 7) this leads to the fact that with an increase of the power of interference owing to the standardizing

action of the AGC system the slope of the discriminational characteristic of the equivalent follow-up system falls.

From the solution of the equation for the complex spectrum of $V(\omega)$ one can determine the spectral density of fluctuations on the output of the receiving mechanism with AGC. It is possible to show that for the stationary random process there takes place the relationship

$$\overline{V(\omega)\,V^*(\omega')} - \overline{V(\omega)}\;\overline{V^*(\omega')} = 2\pi S_v(\omega)\,\delta(\omega-\omega'), \qquad (2.7.18)$$

where $V^*(\omega)$ is the magnitude adjoint with $V(\omega)$;

$S_v(\omega)$ is the spectral density of fluctuations on the output of the receiver.

Substituting the expressions for $V_0(\omega)$ and $V_1(\omega)$ into (2.7.18), in the first approximation we obtain the following expression for the spectral density of the process on the output of the receiver:

$$S_2(\omega) = S_u(\omega)\,|H_0(i\omega)|^2. \qquad (2.7.19)$$

Consequently, in the first approximation the AGC system is equivalent to a linear system with a frequency response determined by the formula (2.7.10). Let us note that the solution for the spectral density of fluctuations on the output of the receiver in the first approximation does not require knowledge of the distributive laws of probabilities of the input perturbation of the AGC system. This solution can be obtained within the framework of the correlation theory.

Let us consider in somewhat more detail what represents the frequency response of a linear system which is equivalent to the AGC system in the first approximation. This is convenient to do by an example of the wide-spread AGC system of the first order the transmission factor of the feedback circuit of which is determined by the formula

$$k_1 H(i\omega) = \frac{k_1}{1 + i\omega T}. \qquad (2.7.20)$$

From formulas (2.7.20) and (2.7.10) we will obtain the following expression

for the square of the equivalent frequency response:

$$|H_э(i\omega)|^2 = \frac{K_{cp}^2 (1+\omega^2 T^2)}{(1+Abk_1)^2 (1+\omega^2 T_э^2)}. \tag{2.7.21}$$

Here

$$T_э = \frac{T}{1+Abk_1} \tag{2.7.22}$$

is the equivalent time constant of the AGC system.

An exemplary form of the equivalent frequency responses of the receiver with AGC is shown in Fig. 2.10. From these characteristics it is clear that the slower the fluctuations, the better they are processed by the AGC system. It is important to note that the equivalent frequency response of a system, to a strong degree, depends on the mean value of the signal A on the input. The higher the mean value of the input signal the wider the frequency band of fluctuations, which are processed by the AGC system. This is a result of the nonlinearity of the system.

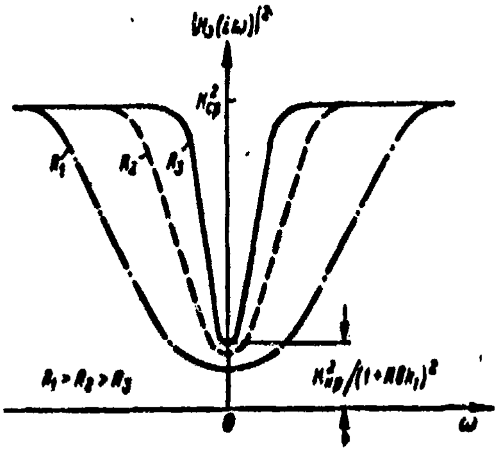

Fig. 2.10. Equivalent frequency responses of a receiver with AGC.

For practical applications it is frequently necessary to know the value of the spectral density of fluctuations on the output with $\omega = 0$. Using the above written relationship for the spectral density and the equivalent frequency response of the system we obtain

$$S_1(0) = \frac{S_a(0) K_{cp}^2}{(1+Abk_1)^2}. \tag{2.7.23}$$

The correction of $S_2(\omega)$ to the first approximation for the spectral density of the fluctuations on the output can be obtained if one were to use formula (2.7.18) and the solution for $V(\omega)$,

$$\overline{V_2(\omega)V_2^*(\omega')}+\overline{V_1(\omega)V_3^*(\omega')}+\overline{V_3(\omega)V_1^*(\omega')}-$$
$$-\overline{V_2(\omega)}\,\overline{V_2^*(\omega')}=2\pi S_2(\omega)\,\delta(\omega-\omega'). \qquad (2.7.24)$$

Here it is taken into consideration that $\overline{V_1(\omega)}=\overline{V_3(\omega)}=0$, since $\overline{U(\omega)}=0$. Substituting the expressions for $V_1(\omega)$, $V_2(\omega)$ and $V_3(\omega)$ into (2.7.24), we will have in the subintegral expression the mathematical expectations of the form

$$\overline{U(x_1)U(x_2)U(x_3)U(x_4)}\,.$$

For the normal distributive law the mathematical expectation of such a form is easy to calculate using the relationship (2.4.23) [33]. Finally, for the correction to the first approximation of the spectral density of fluctuations on the output of the receiver we will receive

$$S_2(\omega)=\frac{b^2\,|H_э(i\omega)|^2}{2\pi K_{cp}^2}\left\{\int_{-\infty}^{\infty}S_u(x)\,S_u(\omega-x)\,[|H_p(ix)|^2+\right.$$
$$+H_p(ix)\,H_p^*(i\omega-ix)]\,dx+2S_u(\omega)\,H_p(0)\times$$
$$\times\int_{-\infty}^{\infty}S_u(x)\,\mathrm{Re}\,[H_p(ix)]\,dx+2S_u(\omega)\times$$
$$\times\int_{-\infty}^{\infty}S_u(x)\,\mathrm{Re}\,[H_p(i\omega+ix)\,H_p(ix)]\,dx+2S_u(\omega)\times$$
$$\left.\times\int_{-\infty}^{\infty}S_u(\omega-x)\,\mathrm{Re}\,[H_p(i\omega)\,H_p(ix)]\,dx\right\}. \qquad (2.7.25)$$

With the help of formulas (2.7.19) and (2.7.25) it is possible to calculate the spectral density on the output of a receiving mechanism with an AGC system of any order with accuracy up to the second approximation inclusively

$$S_в(\omega)\approx S_1(\omega)+S_2(\omega). \qquad (2.7.26)$$

The analysis shows that the successive approximations rapidly converage if the inequality $\frac{\sigma^2}{A^2}<1$, is fulfilled; therefore, for the majority of cases the

accuracy with which is determined the magnitude of the spectral density in the second approximation, appears to be fully sufficient and it is frequently possible to limit ourselves to the first order of approximation.

Dispersion of fluctuations on the output can be found by the integration of the expression for spectral density

$$\sigma_v^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_v(\omega)\,d\omega. \qquad (2.7.27)$$

Thus, if the function of correlation of fluctuations on the input has form of $R_u(\tau) = \sigma^2 e^{-\alpha|\tau|}$, where $\frac{1}{\alpha} = T_{\text{к}}$ is the time of correlation, then its corresponding spectral density has the form

$$S_u(\omega) = \frac{2\alpha\sigma^2}{\alpha^2+\omega^2}. \qquad (2.7.28)$$

Using the above obtained relationship, after integration and necessary conversions we find the first approximation for the dispersion of fluctuations on the output of the receiver with an AGC system of the first order

$$\sigma_1^2 = K_{\text{ср}}^2\sigma^2\left[\frac{\alpha T_{\text{э}}}{1+\alpha T_{\text{э}}} + \frac{1}{(1+Abk_{\text{д}})^2(1+\alpha T_{\text{э}})}\right]. \qquad (2.7.29)$$

From (2.7.29) it is clear that the dispersion of fluctuations on the output depends on the mathematical expectation of the signal on the input. This is a result of the fact that the AGC possesses a variable parameter (amplification factor) controlled by input influence.

If the equivalent time constant of the AGC system is much longer than the time of correlation of the random process on the input, which corresponds to the case of wide-band fluctuations on the input as compared with the frequency band constituents effectively processed by AGC, then the AGC system does not practically process the fluctuations. In this case $\sigma_v^2 \approx K_{\text{ср}}^2\sigma^2$ and the relation of dispersion to the square of the mathematical expectation of the process on the output is approximately equal to their relation on the input, i.e., the receiving mechanism strengthens the random process on the input by $K_{\text{ср}}$ times without

changes. Physically this happens owing to the fact that at large relations of $T_з/T_к$ the dispersion of adjustment voltage is near to zero and the amplification factor of the variable amplifier depends only on the mathematical expectation of this voltage. It is clear that the same position will take place in the case where the AGC system works on wide-band noise interference or on set noises of the receiving mechanism.

2.7.2. Influence of a Signal Fluctuating in Amplitude and Modulated by Sinusoidal Law on a Receiving Mechanism From AGC

In radar stations which use the method of conical scanning the signal reflected from the target fluctuates in amplitude and, owing to the scanning of the receiver antenna, is additionally modulated by sinusoidal law. Since in the input signal envelope information is contained about the angular position of the target, great practical interest is represented by the finding of the characteristics of the signal envelope on the output of the receiving mechanism with AGC. Theoretically this problem is rather complicated since we are dealing with the influence of a non-stationary random process on a nonlinear system.

In connection with the periodicity of the law of modulation the mathematical expectation and dispersion of the random process on the input will be the periodic functions of time.

Using the already introduced idealizations with respect to the characteristics of AGC this problem can be completely solved the same way as this was done during a stationary random process on the input.

In the considered case the perturbation on the input can be written in the form

$$u(t)=A(1+\xi(t))(1+m\cos\Omega_{вх}t)=A+Am\cos\Omega_{вх}t+ \\ +A\xi(t)+Am\xi(t)\cos\Omega_{вх}t, \qquad (2.7.30)$$

where $A\xi(t)$ is the stationary random process with zero mathematical expectation and dispersion of σ^2 ;

A and m is the mean value and modulation percentage of the normal signal;

$\Omega_{\text{ск}}$ is the scanning frequency.

It is not difficult to show that the spectrum of the variable constituents of the signal on the input will be equal to

$$U(\omega)=U_1(\omega)+Am\pi[\delta(\omega+\Omega_{\text{ск}})+\delta(\omega-\Omega_{\text{ск}})]+ \\ +\frac{m}{2}[U_1(\omega+\Omega_{\text{ск}})+U_1(\omega-\Omega_{\text{ск}})], \tag{2.7.31}$$

where $U_1(\omega)$ is the spectrum of the process $A\xi(t)$.

Then the AGC system will be described as before by the integral equation (2.7.6), where $U(\omega)$ is expressed by formula (2.7.31). Let us find the solution of this equation by the method of successive approximations considering the depth of modulation ($m<1$) small and the relation of dispersion to the square of the mean value of the input signal $\left(\frac{\sigma^2}{A^2}<1\right)$. For that it is sufficient to place $U(\omega)$ into the expressions for $V_0(\omega), V_1(\omega), V_2(\omega), V_3(\omega), \ldots$ obtained in par. 2.7.1 in the form of (2.7.31). Found in this way, the solution gives the possibility to determine the statistical characteristics of the signal on the output.

The mathematical expectation will occur if in formula (2.7.15) the resolution for $V(\omega)$ is substituted. Then after the necessary conversions with accuracy up to the second approximation, the mathematical expectation of the signal on the output of the receiver will have the form

$$\overline{v(t)}=\Big\{K_{\text{ср}}A-\frac{b}{2\pi(1+Abk_1)}\int\limits_{-\infty}^{\infty}S_{\Phi}(\omega)H_p(i\omega)\,d\omega- \\ -\frac{bA^2m^2}{2(1+Abk_1)}\operatorname{Re}[H_p(i\Omega_{\text{ск}})]\Big\}+Am\operatorname{Re}\left[H_0(i\Omega_{\text{ск}})e^{i\Omega_{\text{ск}}t}\right]+ \\ +\frac{bA^2m^2}{2K_{\text{ср}}}\operatorname{Re}\left[H_0(2i\Omega_{\text{ск}})H_p(i\Omega_{\text{ск}})e^{2i\Omega_{\text{ск}}t}\right]. \tag{2.7.32}$$

where $S_{\Phi}(\omega)$ is the spectral density of the fluctuations of $A\xi(t)$.

Thus, with accuracy, up to the second approximation the mathematical expectation of the signal on the output contains a constant constituent, the first and second harmonic of the normal signal envelope on the input.

Considering that the circuits located in the channel for separating the angular error signal the second harmonic of the normal signal envelope not gated, henceforth, it will not interest us. Since usually $Abk_1 \gg 1$, then, with a degree of accuracy sufficient for practice, the mathematical expectation of the signal on the output can be recorded in the form

$$\overline{\sigma(t)} \approx K_{\text{ср}} A + Am \mathrm{Re}\left[H_{э}(i\Omega_{\text{ск}})\, e^{i\Omega_{\text{ск}} t}\right]. \qquad (2.7.33)$$

This formula describes the signal on the output when a nonfluctuating signal with a sinusoidal envelope acts on the input of the receiving mechanism from AGC (see for example, [31]).

Consequently, it can be said that with the assumptions made, the fluctuations do not change the mathematical expectation of the signal on the output of the receiving mechanism with AGC.

In view of the nonstationary character of the random process on the input which is introduced owing to the modulation of the fluctuating signal by sinusoidal law, the functions of correlation of the random process on input as well as on the output will depend on the current time. However, on the output of the receiving mechanism in the channel for separating the signal of angular error there are usually put narrow-band mechanisms neutralizing the random process in time. Therefore, a sufficient characteristic of the random process on the output of the receiving mechanism will be the function of the correlation neutralized in time or its corresponding spectral density.

It is not difficult to find that the spectral density of the random process on the input will be expressed by the formula

$$S_{u}'(\omega) = S_{\phi}(\omega) + \frac{m^2}{4}\left[S_{\phi}(\omega - \Omega_{\text{ск}}) + S_{\phi}(\omega + \Omega_{\text{ск}})\right]. \qquad (2.7.34)$$

The form of the spectral density for a narrow-band spectrum of fluctuations on the input is shown in Fig. 2.11.

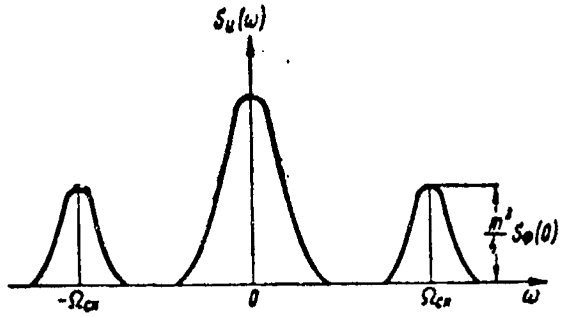

Fig. 2.11. Spectral density of the random process on the input of a receiver.

As can be seen from the figure, in the signal on the input, owing to the modulation of it by sinusoidal law, there already appear constituents of the fluctuations near the scanning frequency $\Omega_{cк}$.

The spectral density corresponding to the mean function in time of the correlation of the process on the output can be determined analogous to the preceding one from the relationship

$$\widetilde{[V(\omega)-\overline{V(\omega)}][V^*(\omega')-\overline{V^*(\omega')}]}=2\pi S_v(\omega)\delta(\omega-\omega'). \qquad (2.7.35)$$

Here the wavy line means that in the mathematical expectation are selected only those members which correspond to the mean function in time of the correlation. Substituting into (2.7.35) the solution for $V(\omega)$, we obtain the first approximation for $S_v(\omega)$ in the form

$$S_v(\omega)=S_\Phi(\omega)|H_\Phi'(i\omega)|^2. \qquad (2.7.36)$$

Thus, in the first approximation the AGC system influences the fluctuation of signal as a linear system with frequency response determined by formula (2.7.10). Analogous to the preceding, it is possible to obtain a correction to the first approximation. A full expression for the magnitude of this correction will not be cited in view of its awkwardness, and there will be used the circumstance that for the channel for separating the angular error signal, only the constituents of fluctuations on the output, which are near the scanning frequency $\Omega_{cк}$, are important. Usually in selecting the parameters of an AGC system, the requirement

of minimum distortions of regular input signal envelope must be considered first. From the formula for the mathematical expectation of the random process on the output of the receiver it is not hard to find that for the satisfaction of this requirement there must be fulfilled the conditions

$$|H_э(i\omega_{ск})|^2 \approx K_{ср}^2, \quad |H_р(i\Omega_{ск})| \approx 0. \tag{2.7.37}$$

In fulfillment of these conditions the constituents of fluctuations on the output near the scanning frequency with a narrow-band spectrum of fluctuations on the output will have the form [34]

$$\Delta S_э(\omega) \approx \frac{m^2}{2} S_\phi(\omega - \Omega_{ск}) |H_э(i\omega - i\Omega_{ск})|^2. \tag{2.7.38}$$

The form of the spectral density of fluctuations on the output of a receiver is shown in Fig. 2.12. From the obtained relationships it follows that

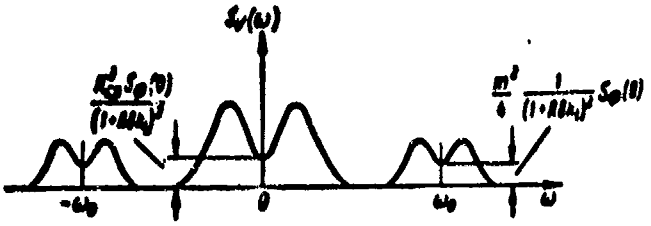

Fig. 2.12. Spectral density of the random process on the output of a receiver with AGC.

the better the AGC system processes the constituents of fluctuations near $\omega = 0$, the better processed are the constituents near the scanning frequency which are in the spectrum of the signal on the input owing to the modulation of the fluctuating signal by sinusoidal law. From this point of view it is profitable to decrease the inertness of the AGC system to the limit determined by the permissible distortions of a sinusoidal signal envelope.

In practical calculations of the noise proof feature of the measurement systems of angular coordinates one should consider that formula (2.7.38) is

obtained on the assumption that the fluctuations on the input near $\omega=0$ are so narrow-band that the constituents of these fluctuations near the scanning frequency are practically equal to zero, i.e., $S_\Phi(\Omega_{ск})\approx 0$. However, this assumption cannot be fulfilled since with a small m the constituents calculated by us $\Delta S_v(\omega)$ can be comparable or even smaller than the constituents of the fluctuations $A_\xi(t)$, near the frequency of $\Omega_{ск}$.

However, these constituents can be calculated using the formula for the first approximation (2.7.36). If the conditions are fulfilled (2.7.37), then part of the spectral density caused by the presence of the considered constituents near the frequency $\Omega_{ск}$, can be considered approximately constant and equal

$$\Delta S(\omega)\approx 2K_{ср}^2 S_\Phi(\Omega_{ск}). \qquad (2.7.39)$$

This means that the AGC system, if the conditions are fulfilled (2.7.37), does not completely process the constituents of the spectrum of fluctuations of the signal on the input of $A_\xi(t)$, which are near the scanning frequency $\Omega_{ск}$. It is possible to show that such appraisals of these constituents of the fluctuations are sufficient.

The obtained characteristics will be limited since they are sufficient for an analysis of the influence of the fluctuations of signal on the radio channel which will be presented in subsequent chapters. More detailed information about the characteristics of the random process on the output of a receiving mechanism with AGC for the considered case and the method with which they are received are contained in [34].

2.7.3. Characteristics of the Fluctuations of a Signal on the Output of a Receiving Mechanism with a Square-Law Detector and an AGC System.

The whole preceding analysis was conducted with the assumption of linearity of the regulated amplifier. With this, the amplifier can include such elements of

a real receiver as an IFA, a detector, and a video amplifier. It is clear that the amplifier can be considered linear only in the case where the detector linearly transmits an input signal envelope, i.e., in the case of a linear detector.

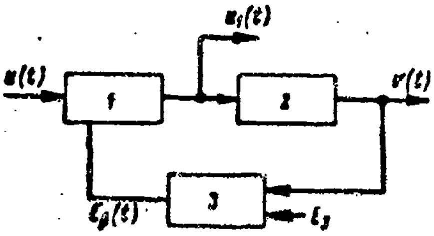

Fig. 2.13. Equivalent diagram of a receiver with a square-law detector and an IFA: 1) intermediate frequency amplifier; 2) square-law detector; 3) feedback circuit.

Since in certain receivers the detector operates on the quadratic portion of its characteristic (with low voltage on the input of the detector), we have a practical interest in the analysis of the influence of the fluctuating signal on such a receiving mechanism with AGC. An equivalent diagram of a receiver with AGC is shown in Fig. 2.13. At first it is convenient to obtain the solution for voltage on the input of the detector $u_1(t)$, and then to find the characteristics of the random process on the output.

Analogous to the preceding (2.7.2) it is possible to write

$$u_1(t) = u(t)[K_0 - bE_p(t)]. \tag{2.7.40}$$

Considering that the voltage on the output of the detector is equal to

$$v(t) = ku_1^2(t), \tag{2.7.41}$$

for the voltage of adjustment we obtain

$$E_p(t) = \frac{kk_1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} U_1(s)U_1(\omega - s)H(i\omega)e^{i\omega t}d\omega ds - k_1E_3, \tag{2.7.42}$$

where $U_1(\omega)$ is the spectrum of $u_1(t)$;

E_3 is the delay voltage converted to the output of the detector.

Substituting (2.7.3) and (2.7.42) into (2.7.40), it is simple to obtain the following integral equation, describing the AGC system in the considered case:

$$U_1(\omega) = A(K_0 + bk_1E_0)2\pi\delta(\omega) + (K_0 + bk_1E_0)U(\omega) - \\ - \frac{Abk_1k}{2\pi}\int_{-\infty}^{\infty} U_1(s)U_1(\omega - s)H(j\omega)\,ds - \\ - \frac{bk_1k}{(2\pi)^2}\iint_{-\infty}^{\infty} U_1(s_1)U_1(s - s_1)H(js)U(\omega - s)\,ds\,ds_1. \tag{2.7.43}$$

We will resolve this nonlinear integral equation by the method of successive approximations with the assumption of the smallness of the relation of dispersion to the square of the mathematical expectation of the signal on the input $\frac{\sigma^2}{A^2} < 1$. Then for a zero approximation we will obtain the following equation:

$$U_{10}(\omega) = A(K_0 + bk_1E_0)2\pi\delta(\omega) - \\ - \frac{Abk_1k}{2\pi}\int_{-\infty}^{\infty} U_{10}(s)U_{10}(\omega - s)H(j\omega)\,ds. \tag{2.7.44}$$

From the physical considerations it is clear that in the zero approximation the signal on the output will be constant. Therefore, it is natural to seek the solution of this equation in the form

$$U_{10}(\omega) = N\delta(\omega)2\pi. \tag{2.7.45}$$

Substituting (2.7.45) into (2.7.44), we will obtain a zero approximation for $U_1(\omega)$

$$U_{10}(\omega) = K_{cp}A2\pi\delta(\omega), \tag{2.7.46}$$

where

$$K_{cp} = \frac{-1 + \sqrt{1 + 4kA^2bk_1(K_0 + bk_1E_0)}}{2kA^2bk_1} \tag{2.7.47}$$

is the average amplification factor of the amplifier.

Substituting the solution for $U_{10}(\omega)$ into the initial integral equation, for the first correction we obtain the following expression:

$$U_{11}(\omega) = \frac{U(\omega)(K_0 + bk_1 E_1)}{1 + 2kA^2 bk_1 K_{cp} H(i\omega)}. \qquad (2.7.48)$$

In the same manner approximations of higher orders can be obtained; however, for practical purposes in most cases it is possible to limit ourselves to the first approximation. Thus, we obtained the solution for the spectrum of a signal envelope on the input on the detector in the form

$$U_1(\omega) \simeq U_{10}(\omega) + U_{11}(\omega).$$

From formula (2.7.41) it follows that for finding the spectrum of the random process on the output of a detector $V(\omega)$ it is sufficient to take the contraction of the spectra

$$V(\omega) = \frac{k}{2\pi} \int_{-\infty}^{\infty} U_1(s) U_1(\omega - s)\, ds \simeq V_0(\omega) + V_1(\omega).$$

Substituting the resolution for $U_1(\omega)$, we obtain

$$V_0(\omega) = K_{cp}^2 kA^2 2\pi\delta(\omega)$$

$$V_1(\omega) = K_{cp} \frac{2U(\omega) K_{cp} kA}{1 + 2K_{cp} kA^2 bk_1 H(i\omega)}.$$

Then the mathematical expectation of the signal on the output will have the form

$$\overline{v(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{V(\omega)} e^{i\omega t} d\omega \simeq kK_{cp}^2 A^2 +$$

$$+ \frac{k}{2\pi} \int_{-\infty}^{\infty} S_u(\omega) |H_{01}(i\omega)|^2 d\omega. \qquad (2.7.49)$$

where

$$H_{01}(i\omega) = \frac{K_{cp}}{1 + 2kA^2 bk_1 K_{cp} H(i\omega)}; \qquad (2.7.50)$$

$S_u(\omega)$ is the spectral density of the fluctuations on the input of the receiver.

For an AGC system with very great inertness it is possible to approximately consider that $|H_{01}(i\omega)|^2 \simeq K_{cp}^2$.

Then

$$\overline{v(t)} \simeq kK_{cp}^2 \overline{u^2(t)}. \qquad (2.7.51)$$

The mean value of the random process on the output remains approximately constant near the magnitude E_3 of delay voltage. Thus, in the case of a square-law detector and the AGC system, with great inertness, the product of the mean amplification factor on the mean square of the input signal envelope is kept constant.

Using the solution for $V(\omega)$, analogous to the preceding for the spectral density of the fluctuations on the output we obtain in the first approximation the following expression:

$$S_1(\omega)=S_u'(\omega)\,|H_э(i\omega)|^2, \tag{2.7.52}$$

where

$$H_э(i\omega)=K_{cp}\frac{2K_{cp}kA}{1+2K_{cp}kA^2bk_1H(i\omega)} \tag{2.7.53}$$

is the frequency response of the linear system which is equivalent to the AGC system in the first approximation

Comparing formulas (2.7.10) and (2.7.53), we see that the equivalent frequency responses of AGC with a linear and square-law detector have the same character, differing only quantitatively.

In conclusion we will present the magnitude of the spectral density of fluctuations on the output at zero frequency, which will be required in subrequent chapters. Substituting in formula (2.7.52) $\omega=0$ and considering that usually magnitude $Abk_1 \gg 1$, we will receive

$$S_v(0)=S_u(0)\frac{K_{cp}^2}{(Abk_1)^2}. \tag{2.7.54}$$

2.7.4. Influence of a Fluctuating Signal on a Double-Loop AGC System

To fulfill all the increasing requirements of receiving mechanisms, in many cases there appears the necessity of application of double-loop AGC systems which can be series (Fig. 2.14, a) and parallel (Fig. 2.14, b).

We will pursue the analysis of the operation of the receiving mechanism with

a double-loop AGC system with an influence on its input of a fluctuating signal. From Fig. 2.14, a it is clear that a receiver with a series double-loop AGC system consists of two series connected amplifiers, each of which is regulated by its own AGC system. An analysis of the random processes in one amplifier with AGC has already been conducted. There remains only a signal with the found characteristics to gate once more one regulated amplifier. With accuracy up to the first approximation this problem is solved elementarily. It appears that a receiver with such a double-loop AGC system acts on the input signal envelope rounding as a linear system with equivalent frequency response determined by the formula

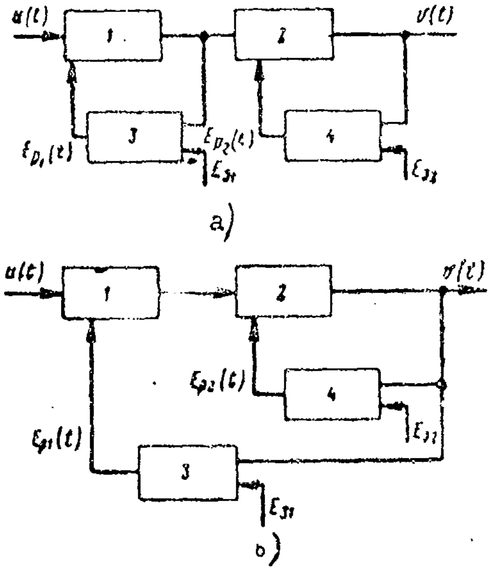

Fig. 2.14. Equivalent diagram of a receiver with a double-loop AGC system:
a) series; b) parallel.
1, 2) variable amplifiers; 3,4) feedback circuit.

$$H_{\text{э}}(i\omega) = H_{\text{э}1}(i\omega) H_{\text{э}2}(i\omega), \qquad (2.7.55)$$

where $H_{\text{э}1}(i\omega)$ and $H_{\text{э}2}(i\omega)$ are the equivalent frequency responses of the first and second amplifier; $H_{\text{э}1}$ is determined by the formula (2.7.10).

In finding $H_{э2}(i\omega)$ in formula (2.7.10) one should replace A by $K_{ср1}A$, where $K_{ср1}$ is the average amplification factor of the first amplifier and all remaining parameters, entering into the formula, relate to the second amplifier.

The analysis of a parallel double-loop AGC system is more complicated. Being interested in the steady-state operation, we approximate the regulating characteristics of amplifiers by straight lines. Then the signal on the output v(t) (by v(t) and u(t) are understood real signal envelopes) will be connected with the input signal and the adjustment voltages in the following manner:

$$v(t)=u(t)[K_1-b_1E_{p1}(t)][K_2-b_2E_{p2}(t)], \tag{2.7.56}$$

where K_1 and b_1, K_2 and b_2 are the parameters of the controlling characteristics of first and second amplifiers.

Representing the signals on the input, and output in the form of (2.7.3) and (2.7.4), and the adjustment voltage in the form of (2.7.5), it is not hard to obtain an integral equation for the complex spectrum $V(\omega)$ of the signal on the output

$$\begin{aligned}V(\omega)=F(\omega)\Big\{&[2\pi A\delta(\omega)+U(\omega)](K_1+b_1k_{01}E_{31})\times\\ &\times(K_2+b_2k_{02}E_{32})-(K_2b_1+b_1b_2k_{02}E_{32})\times\\ &\times\frac{k_{01}}{2\pi}\int\limits_{-\infty}^{\infty}V(s)H_1(is)U(\omega-s)\,ds-(K_1b_2+b_1b_2k_{01}E_{31})\times\\ &\times\frac{k_{02}}{2\pi}\int\limits_{-\infty}^{\infty}V(s)H_2(is)U(\omega-s)\,ds+\\ &+\frac{Ab_1b_2k_{01}k_{02}}{2\pi}\int\limits_{-\infty}^{\infty}V(s)V(\omega-s)H_1(is)H_2(i\omega-is)\,ds+\\ &+\frac{b_1b_2k_{01}k_{02}}{(2\pi)^2}\iint\limits_{-\infty}^{\infty}V(x)V(s-x)H_1(ix)H_2(is-ix)\times\\ &\times U(\omega-s)\,dx\,ds\Big\},\end{aligned} \tag{2.7.57}$$

where

$$F(\omega)=\frac{1}{1+(Ab_1K_2k_{01}+Ab_1b_2k_{01}k_{02}E_{32})H_1(i\omega)+(Ab_2K_1k_{02}+Ab_1b_2k_{01}k_{02}E_{31})H_2(i\omega)};$$

$k_{01}H_1(i\omega)$ and $k_{02}H_2(i\omega)$ are the frequency responses of the AGC feedback circuit.

The solution of this nonlinear integral equation is expediently sought as before in the form of consecutive approximations

$$V(\omega)=V_0(\omega)+V_1(\omega)+\dots \qquad (2.7.58)$$

Not remaining in detail on finding this solution we will present the final results [35]

$$V_0(\omega)=2\pi K_{cp}A\delta(\omega), \qquad (2.7.59)$$

where $K_{cp}=\frac{D}{A}$ is the mean amplification factor of the receiving mechanism, and D is determined from the solution of the equation

$$\begin{gathered}b_1b_2k_{01}k_{02}AD^2-(1+Ab_1K_2k_{01}+Ab_2K_1k_{02}+Ab_1b_2E_{21}k_{01}k_{02}+\\+Ab_1b_2E_{12}k_{01}k_{02})D+A(K_1K_2+b_1K_2k_{01}E_{11}+\\+b_2K_1E_{22}k_{02}+b_1b_2E_{12}k_{01}k_{02})=0.\end{gathered}$$

The magnitude K_{cp} coincides with the amplification factor of the receiving mechanism when on its input there acts a signal of constant amplitude equal to A.

The first correction to the zero approximation has the form

$$V_1(\omega)=U(\omega)H_0(i\omega), \qquad (2.7.60)$$

$$\begin{gathered}H_0(i\omega)=\frac{K_{cp}}{1+N_1H_1(i\omega)+N_2H_2(i\omega)},\\N_1=Ab_1k_{01}[K_2-b_2k_{02}(K_{cp}A-E_{22})],\\N_2=Ab_2k_{02}[K_1-b_1k_{01}(K_{cp}A-E_{11})].\end{gathered} \qquad (2.7.61)$$

From these formulas it follows that from the viewpoint of transmission of the signal envelope the receiving mechanism with a double-loop AGC system is equivalent in the first approximation to a linear system with frequency response $H_0(i\omega)$. Using the obtained solution one can determine the spectrum of the signal envelope on the output in any form of an envelope on the input.

With a random influence on the input the obtained solution permits finding

the characteristic of the random process on the output. The mathematical expectation of the random process on the output in the first approximation will equal to

$$\overline{V(t)} \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} [\overline{V_0(\omega)} + \overline{V_1(\omega)}] e^{i\omega t} d\omega = K_{cp} A. \qquad (2.7.62)$$

The spectral density of the fluctuations on the output is simple to determine by using the relationship (2.7.18). The first approximation for the unknown spectral density will have the form

$$S_1(\omega) = S_u(\omega) |H_э(i\omega)|^2. \qquad (2.7.63)$$

Analogous to the preceding, these formulas for the mathematical expectation and spectral density of the fluctuations on the output are correct with any form of the distributive law of probabilities of the input random process envelope.

The obtained relationships permit finding the characteristic of the signal envelope on the output of the receiver and to estimate the advantage of application of the double-loop AGC system. For example, in radar sets with conical scanning the goniometric channel of the receiving mechanism should have an AGC system with great inertness. In the range channel, on the other heand, it is more desirable to demodulate the signal, i.e., to have a low-inertial AGC. These requirements can be fulfilled with success by employing the double-loop AGC system with any of the above described forms if the first loop is made inert, and the second—low-inertial. It is interesting to note an additional advantage which is obtained owing to the application of the double-loop AGC system. The demodulating properties of AGC in the range channel will depend weakly on the level of the signal on the input. Actually, from the formulas (2.7.55) and (2.7.61) it follows that within the limit, if the first loop of the AGC possesses an infinitely large inertness the equivalent frequency response of the receiver in both cases has the form

$$H_э(i\omega) \approx \frac{K_{cp}}{1 + K_{cp1} A b_1 k_{o2} H_2(i\omega)}.$$

Magnitude $K_{\text{cp1}}A$ remains approximately constant with a change of the mean level of the input signal A in a large range, and signifies the frequency response on which the demodulating properties of the AGC depend.

Let us remember that in the case of a simple single-loop AGC system the frequency band of the signal envelope processed by the AGC was increased with the growth of the mean level of input signal.

In conclusion let us note that a similar method in [35] analyzes the action on a double-loop AGC system at a signal with a normal sinusoidal envelope.

2.7.5. Influence of Random Pulse Interference

Let us consider the influence on a receiver mechanism with AGC of random pulse interference (see Ch. 1). One should always expect the power of interference to be significantly larger than the power of a useful signal. Depending upon the magnitude of the mean frequency of the appearance of interference pulses the following operating modes of a receiver mechanism with AGC are possible.

1. The mean frequency of the pulse interference is small. On the output of the receiver widely spaced pulses of interference will take place with an amplitude equal to the level of the limitation of the receiver mechanism and the pulses of a useful signal (Fig. 2.15,a). Owing to the influence of interference on the output of the AGC feedback circuit there appears a random constituent of adjustment voltage which, acting on the amplification factor of the receiver, modulates the pulses of the useful signal in a random manner. Since in this mode the useful signal is higher than the delay level, the AGC system will partially process the random constituents of the signal envelope. The quantitative characteristics interesting us of the random process on the output for that case will be found below.

2. The mean frequency of the appearance of interference pulses is increased. The mean value of the AGC adjustment voltage increases at a certain critical frequency $\nu_{\text{кр1}}$ and the useful signal will drop below the delay level (Fig. 2.15,b).

At this moment the AGC system ceases being closed to the useful signal and this means it does not process the random constituent of the signal envelope appearing due to the interference.

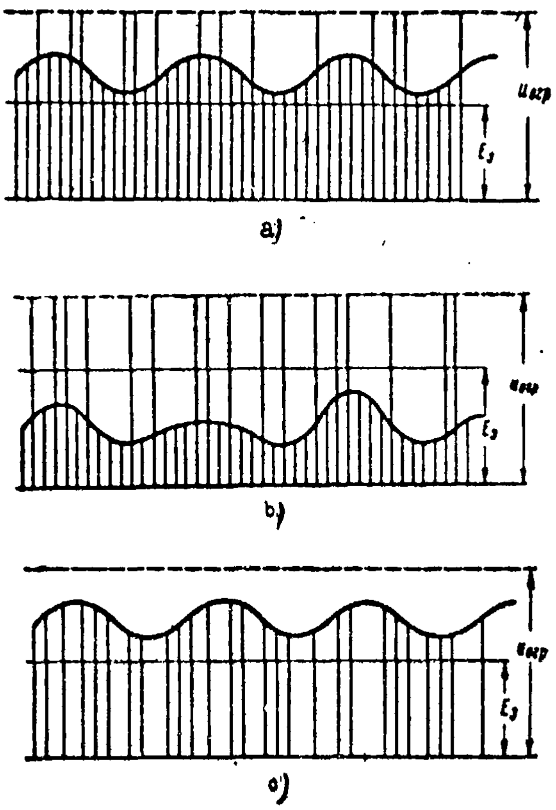

Fig. 2.15. The influence of random pulse interference on a receiver with AGC.

3. The mean frequency of the interference pulses is increased, still more then the amplification factor of the receiving mechanism decreases so much that the useful signal appears completely suppressed, and at a certain frequency of $\nu_{кр2}$ the interference emerges from under the limitation (Fig. 2.15,c).

Such is the physics of the influence of the random pulse interference on a receiver mechanism with AGC. Let us consider the quantitative characteristics of the operation of a receiver mechanism in the presence of such an interference.

With a sufficiently widely spaced interference, where the useful signal is

higher than the delay level, for an analysis of radar meters it is necessary to know the characteristic of the signal envelope on the output having, as was indicated above, a random character. We will define these characteristics. The adjustment voltage in the considered case can be recorded in the form

$$E_{\text{р}}(t)=E_{\text{рс}}(t)+\overline{E(t)}+e_{\text{рп}}(t), \tag{2.7.64}$$

where $E_{\text{рс}}(t)$ is the adjustment voltage appearing due to the useful signal and determined by formula (2.7.5);

$\overline{E(t)}$ is the mean value of the constituent of adjustment voltage appearing due to interference;

$e_{\text{рп}}(t)$ is the random constituent of the adjustment voltage appearing due to interference $(\overline{e_{\text{рп}}(t)}=0)$.

Representing the adjustment voltage in the form of a Fourier integral we obtain

$$E_{\text{р}}(t)=\frac{k_1}{2\pi}\int_{-\infty}^{\infty}V(\omega)H(i\omega)e^{i\omega t}d\omega-k_1E_{\text{з}}+$$
$$+\overline{E(t)}+\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{\text{рп}}(\omega)e^{i\omega t}d\omega, \tag{2.7.65}$$

where $E_{\text{рп}}(\omega)$ is the spectrum of $e_{\text{рп}}(t)$.

Putting (2.7.65) into (2.7.2) and taking into account (2.7.3) and (2.7.4), we obtain an integral equation describing the AGC system:

$$V(\omega)=\frac{1}{1+Abk_1H(i\omega)}\Big\{2\pi A(K_0+bk_1E_{\text{з}}-b\overline{E(t)})(\delta(\omega)+$$
$$+U(\omega)(K_0+bk_1E_{\text{з}}-b\overline{E(t)})-AbE_{\text{рп}}(\omega)-$$
$$-\frac{bk_1}{2\pi}\int_{-\infty}^{\infty}V(s)U(\omega-s)H(is)ds-$$
$$-\frac{b}{2\pi}\int_{-\infty}^{\infty}U(\omega-s)E_{\text{рп}}(s)ds\Big\}.$$

We will solve this equation by the method of successive approximations with the assumption of the smallness of $\frac{\sigma^2}{A^2}<1$ and $\frac{\sigma_{\text{р}}^2}{A^2}<1$, where $\sigma_{\text{р}}^2$ is the dispersion of the constituent of the adjustment voltage appearing due to interference.

Then the successive approximations will be written in the form

$$V_0(\omega) = 2\pi K_{\text{ср}} A \delta(\omega),$$

where

$$\left.\begin{aligned} K_{\text{ср}} &= \frac{K_0 + bk_1 E_{\text{з}} - b\overline{E(t)}}{1 + Abk_1}; \\ V_1(\omega) &= H_0(i\omega)\left\{U(\omega) - \frac{Ab}{K_{\text{ср}}} E_{\text{рп}}(\omega)\right\}; \\ H_0(i\omega) &= \frac{K_{\text{ср}}}{1 + Abk_1 H(i\omega)}. \end{aligned}\right\} \tag{2.7.66}$$

Approximations of higher orders exist in a similar manner. We will not cite them in view of the cumbersomeness of the expressions.

As an example let us consider the receiver of a radar set with conical scanning. The useful signal in it will be modulated by sinusoidal law with a scanning frequency of $\Omega_{\text{ск}}$, and in the obtained formulas, instead of $U(\omega)$, one should place the expression

$$U(\omega) = mA\pi[\delta(\omega + \Omega_{\text{ск}}) + \delta(\omega - \Omega_{\text{ск}})]. \tag{2.7.67}$$

Then, analogous to the preceding, one can determine the first approximation for the spectral density of the signal envelope on the output

$$S_1(\omega) = S_{\text{рп}}(\omega)\frac{A^2 b^2}{K_{\text{ср}}^2}|H_0(i\omega)|^2, \tag{2.7.68}$$

where $S_{\text{рп}}(\omega)$ is the spectral density of the constituent of the adjustment voltage appearing due to the interference.

The first correction to this approximation can be found from the relationship (2.7.24).

For goniometric channels only the constituents near the scanning frequency will interest us. Substituting the solution for $V_0(\omega)$ and $V_1(\omega)$ into (2.7.24), we obtain that the correction to the first approximation contains the following interference constituents near the frequency of scanning:

$$\Delta S(\omega)\approx\frac{1}{2}S_{\text{рп}}(\omega-\Omega_{\text{ск}})\left(\frac{Amb}{K_{\text{ср}}}\right)^{2}|H_{\text{э}}(i\omega)|^{2}\left\{\left(\frac{Ab}{K_{\text{ср}}}\right)^{2}\times\right.$$
$$\left.\times|H_{\text{р}}(\omega-\Omega_{\text{ск}})|^{2}-2\frac{Ab}{K_{\text{ср}}}\operatorname{Re}[H_{\text{р}}(\omega-\Omega_{\text{ск}})]+1\right\}. \qquad (2.7.69)$$

Let us turn to a determination of the spectral density $S_{\text{рп}}(\omega)$. Since the receiver mechanism is gated, the pulses of the signal or interference will appear on the output only at the moments of the gating. If the interference conforms to the Poisson distribution then the probability of the appearance of an interference pulse in the strobe will be equal to

$$p=\nu\tau_{\text{с}},$$

where ν is the mean interference frequency;

$\tau_{\text{с}}$ is the duration of the strobe; $p < 1$.

Then the disperion of amplitude of the interference impulse on the input of the feedback circuit can be written in the form

$$\sigma_{\text{п}}^{2}\approx(u_{\text{огр}}-E_{\text{з}})^{2}\nu\tau_{\text{с}}(1-\nu\tau_{\text{с}}), \qquad (2.7.70)$$

where $u_{\text{огр}}$ is the limitation voltage of the receiver.

The interference can be represented in the form of pulses following with the frequency of gating repetition where the dispersion of the amplitude of every pulse is determined by the formula (2.7.70), and the adjacent pulses, naturally, can be considered independent. Since in feedback circuits there is a filter with a time constant many times larger than the period of gating repetition, the filter will react to such an interference as to white noise with a spectral density determined from the relationship

$$N_{\text{п}}=\sigma_{\text{п}}^{2}T_{r},$$

where T_r is the period of gating repetition.

Then on the output of the AGC feedback circuit the spectral density of the constituent of the interference will have the form

$$S_{\text{рп}}(\omega)=\sigma_{\text{п}}^{2}T_{r}k_{1}^{2}|H(i\omega)|^{2} \qquad (2.7.71)$$

or, considering (2.7.70), we find

$$S_{p\text{п}}(\omega)=(u_{\text{огр}}-E_з)^2\nu\tau_c(1-\nu\tau_c)T_r k_1^2|H(i\omega)|^2. \qquad (2.7.72)$$

The obtained formulas permit calculation of the spectral density of the constituents of the interference near the scanning frequency. In the analysis of the goniometrical channels it is frequently sufficient to know only the value of this spectral density on the scanning frequency. Let us consider that usually the parameters of the AGC are selected in this way so that distortions of the sinusoidal signal envelope are minimum.

Here

$$|H_э(i\Omega_{\text{ск}})|^2\approx K^2_{\text{ср}}.$$

Then from formulas (2.7.68), (2.7.69) and (2.7.72), considering that magnitude $Abk_1 \gg 1$, for the spectral density of the constituents of the interference of the signal envelope on the scanning frequency we obtain the following simple expression:

$$S_v(\Omega_{\text{ск}})\approx(u_{\text{огр}}-E_з)^2\nu\tau_c(1-\nu\tau_c)T_r\times$$
$$\times\left[\frac{m^2}{2}+2(Abk_1)^2|H(i\Omega_{\text{ск}})|^2\right]. \qquad (2.7.73)$$

In the considered mode, the receiver mechanism will operate as long as the mean frequency does not exceed a certain value of $\nu_{\text{кр}1}$, with which the useful signal will drop lower than the delay level. We will define this interference frequency.

It is possible to approximately obtain that the transition to a new operating mode will be carried out while the mean value of the useful signal on the output is equal to the delay voltage, i.e., at the fulfillment of the equality

$$\overline{\overline{v(t)}}=A(K_0-b\overline{E_p(t)})=E_з. \qquad (2.7.74)$$

In this mode the adjustment voltage will be determined only by the interference since the useful signal is lower than the delay level. The mean value of the

adjustment voltage is simple to find by the formula

$$\overline{E_{\text{р}}(t)} = (u_{\text{огр}} - E_{\text{з}})\, k_1 \nu \tau_{\text{c}}. \tag{2.7.75}$$

Substituting (2.7.75) into (2.7.74) and solving relative to ν, we obtain the following expression:

$$\nu_{\text{кр1}} = \frac{K_0 A - E_{\text{з}}}{\tau_c A b k_1 (u_{\text{огр}} - E_{\text{з}})}. \tag{2.7.76}$$

With this, the parameters of the regulating characteristic of K_0 and b should be selected on the section corresponding to the normal operation with a signal on the input with amplitude A, equal to the amplitude of the useful signal.

Thus, if the mean interference frequency is higher than $\nu_{\text{кр1}}$, the signal will drop below the delay level, and its mean value will be determined by the formula

$$\overline{\overline{v(t)}} = A\,[K_0 - b k_1 \nu \tau_c (u_{\text{огр}} - E_{\text{з}})] = K_{\text{ср}} A. \tag{2.7.77}$$

The adjustment voltage can be written in the form

$$E_{\text{р}}(t) = \overline{E(t)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\text{рп}}(\omega)\, e^{i\omega t} d\omega. \tag{2.7.78}$$

Substituting (2.7.78) into (2.7.2) and considering (2.7.3) and (2.7.4), for the spectrum of the signal envelope on the output we obtain the following expression:

$$V(\omega) = K_{\text{ср}}\,[2\pi A \delta(\omega) + U(\omega)] - A b E_{\text{рп}}(\omega) - \\ - \frac{Ab}{2\pi} \int_{-\infty}^{\infty} U(s)\, E_{\text{рп}}(\omega - s)\, ds.$$

Using this expression it is easy to find the spectral density of the signal envelope on the output for the above considered case of a radar set with conical scanning. Then the constituents of interference near the frequency of scanning will be determined from the formula

$$\Delta S(\omega) = 2A^2 b^2 S_{\text{рп}}(\omega) + \frac{A^2 b^2 m^2}{2} S_{\text{рп}}(\omega - \Omega_{\text{ск}}).$$

The value of the spectral density of the signal envelope on the scanning

frequency, taking into account (2.7.78), will be equal to

$$S_v(\Omega_{\text{ск}}) = (U_{\text{огр}} - E_з)^2 \nu\tau_c (1 - \nu\tau_c) T_r (Abk_1)^2 \times \times \left\{ 2|H(i\Omega_{\text{ск}})|^2 + \frac{m^2}{2} \right\}. \qquad (2.7.79)$$

It is not difficult to see that in the analyzed mode the value of the spectral density on the scanning frequency is larger than its value in the case where the AGC system is closed to the useful signal. This should be expected, since in the considered mode the AGC system does not process random constituents of the signal envelope.

In conclusion, let us find the value of the mean frequency of interference $\nu_{\text{кр}2}$, with which the interference emerges from under the limitation. The transition to such an operating mode will take place at fulfillment of the equality

$$u_п (K_0 - b\overline{E_p(t)}) = u_{\text{огр}}, \qquad (2.7.80)$$

where $u_п$ is the amplitude of interference on the input of the receiver.

Substituting the mean value of the adjustment voltage determined by formula (2.7.75), and solving the equation relative to ν, we obtain

$$\nu_{\text{кр}2} = \frac{K_0 u_п - u_{\text{огр}}}{\tau_c b k_1 (u_{\text{огр}} - E_з)}. \qquad (2.7.81)$$

Let us note that the parameters of the approximated regulating characteristic of the receiver in the given case should be selected at the operating point where the amplification factor is equal to

$$K = \frac{u_{\text{огр}}}{u_п}.$$

The found characteristics of the random process on the output and the values of the critical frequencies of interference, with which the transition from one operating mode to another occurs, is sufficient for an analysis of the influence of such interference on the radio channel on the whole.

2.7.6. Influence of Intermittent Interference

In Chapter 1 it was noted that interference can have an intermittent character. Let us consider the processes which occur in the receiving mechanism with AGC with the influence of interference of such a form. Interference will be considered active in the time interval $T_{\text{вкл}}$ and cutoff for the time $T_{\text{выкл}}$ (Fig. 2.16).

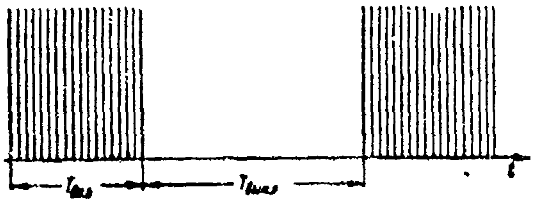

Fig. 2.16. Intermittent interference.

The power of the interference is naturally expected much higher than the power of the useful signal as a consequence of which at the first moments of time the interference occurs under the limitation in the receiver. Then on the input of the AGC feedback circuit there will act a voltage equal to $u_{\text{огр}} - E_{\text{з}}$. Owing to this there will be produced an adjustment voltage decreasing the amplification factor of the receiver. In view of great inertness of the AGC feedback circuit the adjustment voltage will increase slowly. Therefore, for the duration of a certain time, which will be designated $t_{\text{вкл}}$, the amplification factor of the receiver will be such that interference remains limited by the amplitude (Fig. 2.17). For the duration of this time the AGC system appears to be opened since, in spite of the fact that the adjustment voltage increases, the interference amplitude on the output remains constant, equal to $u_{\text{огр}}$.

Lastly, the moment approaches when the increasing adjustment voltage will change the amplification factor so much that the interference will emerge from under the limitation. From this moment the AGC cuts off and starts a fast transient process at the end of which the amplitude of the interference on the output attains a steady value.

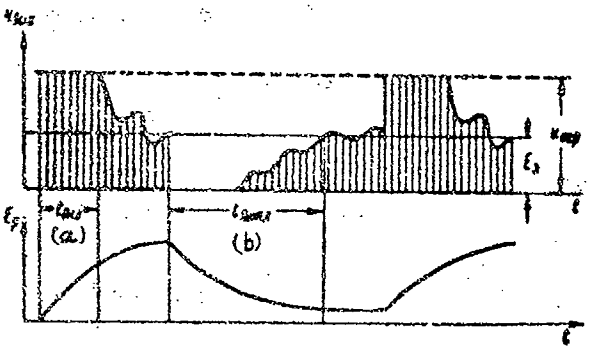

Fig. 2.17. Intermittent interference and a useful signal on the output of a receiver. KEY: (a) Tactive; (b) T cut-off.

After the start of the bundle of interference on the output of the AGC filter the voltage remains equal to the steady value of the adjustment voltage during operation on the interference. Since the amplitude of the interference is much larger than the amplitude of the useful signal then in the first moments of time after cut-off of the interference the useful signal appears to be completely suppressed. Then the capacitors of the AGC filter start to discharge, the adjustment voltage decreases, owing to which the amplification factor of the receiver increases. After a certain time $t_{\text{выкл}}$ the useful signal increases to the delay level and the AGC system starts to operate on the useful signal. With the reception of the following bundle of interference the described cycle of operation is repeated.

The described form of interference leads to the fact that during the time $t_{\text{вкл}}$, while the interference is under the limitation, and $t_{\text{выкл}}$, while the useful signal is absent or is very small, on the output of the receiver there is no information about the target position data. Furthermore, in radar sets with conical scanning beats appear between the harmonics of the breaking frequency and the reference voltage in the goniometric channels, as a result of which tracking by angles can be completely affected.

The important quantitative characteristics of the operation of a receiving mechanism in the presence of such an interference are introduced above the magnitude of $t_{\text{вкл}}$ and $t_{\text{выкл}}$. Their determination will be limited during

the analysis.

If the filter in feedback circuits is a simple RC-circuit with a time constant T_Φ, then the magnitude $t_{\text{вкл}}$ can be determined by the formula

$$t_{\text{вкл}} = T_\Phi \ln \frac{k_1 (u_{\text{огр}} - E_э)}{k_1 (u_{\text{огр}} - E_э) - \frac{K_0 u_п - u_{\text{огр}}}{b u_п}}, \tag{2.7.82}$$

where K_0 and b are the parameters of the approximated controlling characteristics which are selected in the operating point where the amplification factor of the receiver is equal to $K = \frac{u_{\text{огр}}}{u_п}$;

$u_п$ is the amplitude of the interference on the input.

If the magnitude of the adjustment voltage, necessary for the removal of interference from under the limitation is much less than $k_1(u_{\text{огр}} - E_э)$, then the formula (2.7.82) takes a simpler form

$$t_{\text{вкл}} \simeq T_\Phi \frac{(K_0 u_п - u_{\text{огр}})}{u_п b k_1 (u_{\text{огр}} - E_э)}. \tag{2.7.83}$$

Magnitude $t_{\text{выкл}}$ is determined by the expression

$$t_{\text{выкл}} = T_\Phi \ln \frac{(K_0 u_п - E_э)}{(K_{01} u_c - E_э)} \frac{u_c b_1 k_1}{(1 + u_п b k_1)}, \tag{2.7.84}$$

where K_{01} and b_1 are determined at the operating point of the controlling characteristic in which the amplification factor is equal to $K = \frac{E_э}{u_c}$.

2.8. Conclusion

In the present chapter there was considered the passage of a signal and interferences through the elements of a radio receiver mechanism. It is very important to further note the idealizations of the elements of the receiver which will be used during the analysis and synthesis of radar meters and systems of detection.

An intermediate frequency amplifier can be considered equivalent to a linear band filter tuned to an intermediate frequency. During a pulse signal,

owing to the application of gating in the IFA time selection is possible. As one will see subsequently, the time selection and filtration before detection are an essential part of the optimum processing of a signal.

The second detector can be considered as a mechanism separating the input envelope of the random process (in the case of a linear detector) or the square of the envelope (in the case of square-law detectors).

The phase detector with quadratic characteristics of diodes is equivalent to a simple mechanism cross-multiplying the input random processes with the subsequent separation of low-frequency constituents. Therefore, henceforth the operation of multiplication of the signals will signify the necessity of the transmission of signals through a phase detector of such a form.

Video amplifiers and audio amplifiers will be considered as linear inertialess quadripoles.

In the present chapter is given a detailed analysis of the influence of a fluctuating signal on an AGC system. This information can be necessary in the analysis of the radio channels of specific radar sets.

Henceforth relative to the AGC system the following idealizations will be used:

— with a linear detector in avariable amplifier the AGC system changes the mean value of the amplification factor of the receiver reciprocal to the magnitude of the mathematical expectation of the random process envelope on the input so that the mean value of the amplitude of the voltage on the output remains approximately constant the (standardizing property of the AGC). In a square-law detector and an AGC system with a high inertness a constant voltage is supported on the output equal to the product of the mean amplification factor of the receiver on the mean square of the random process envelope on the input;

— the fluctuations are processed by the AGC system in the first approximation as an equivalent linear system with frequency responses determined by the formulas (2.7.10) or (2.7.53).

C H A P T E R 3

GENERAL QUESTIONS OF THE THEORY OF DETECTION

3.1. Introductory Remarks

Detection of targets is one of the most important problems of radar. In radar sets of remote detection, this problem is sometimes unique. In an impressive majority of radar systems, detection precedes the fulfillment of other problems--identification of targets, accurate determination of their coordinates, explanation of the character of trajectories of detected targets, etc.

In some radar stations, the problem of detection in pure form, as a problem of establishment of the presence of a target, does not appear, since these radar sets receive an indication on the presence of a target in a certain region of space from special radar sets of detection. However, the accuracy, with which the position of the target relative to the considered radar set is determined with the help of a station of detection (accuracy of target designation), practically always is insufficient for transition to accurate determination of current coordinates and identification of targets with the help of specially intended tracking systems. Therefore, the fulfillment, by such radar, of its main functions, should precede the capture by the tracking system, which is a more precise definition of the target position data. This more precise definition is produced technically by the same means as detection of the target, and appearing with this, the theoretical problems of its formulation and methods of

solution in most cases are fully analogous to the problems of detection.

It is necessary to note that the problem of determination of coordinates of the detected target always arises and is solved simultaneously with the problem of detection and is naturally connected with it. Separation of these problems can be produced only very conditionally and is connected mainly with the essential distinction of technical methods of determination of coordinates in conditions of detection and measurement, where the methods, utilized in conditions of detection, are closely connected with the procedure of acceptance of the solution on the presence of the target.

For a comparison of various systems of detection, it is necessary to introduce definite quantitative characteristics of the quality of these systems. The characteristics of authenticity of the accepted solution on the presence or absence of the target can be, for example, the probability of pass of the target (acceptance of solution on the absence of the target, when there is a target) and false alarm (acceptance of solution on the presence of target, when there is no target) or any function of these probabilities. The characteristics of accuracy of determination of coordinates, producible in conditions of detection, can be statistical characteristics of errors of measurement; for example, dispersion or probability of exceeding by error in modulus of the half-width of the discriminational characteristic corresponding to the tracking system. The character of requirements, presented to systems of detection, and selection of quantitative characteristics of the quality of their work is determined by the specific conditions of application of these systems and the essence of the considered tactical problem.

If one were limited to the consideration of existing systems of detection, then in this region, the problems of the theory reduce to the calculation and comparison of quantitative characteristics of various systems and rational selection of their parameters. Such a problem may be transferred to problems of

analysis of systems of detection.

The contemporary state of the theory does not limit us to the analysis and comparison of systems of detection, designed, proceeding from those or other qualitative considerations. Application to problems of detection, quite in detail developed in mathematics of the theory of statistical solutions (the elements of this theory are expounded in Section 3.2) allows to synthesize forms of treatments of the received signal, ensuring the best possible given conditions of the value of the selected characteristics of quality.

In the theory of statistical solutions are considered problems about optimum (from the viewpoint of selected criteria of quality) determination of probability properties of random processes or totalities of random variables by the results of observation of realizations of these processes or magnitudes. A problem of such type is also a problem of detection: on the basis of observation of realization of the received signal, which is random due to the interferences and fluctuations of the reflected signal, it is required to accept the solution on the presence of the target, i.e., to accept the solution on, which of the two possible distributive laws (for a signal with interference or for one interference) is subordinated to a random process, the realization of which is observed.

Although the identity of problems of detection and some problems, considered in the theory of statistical solutions, is quite evident, wide application of the results of the theory to problems of detection in radar began comparatively recently after the appearance of the already existing classical works of V. Peterson, T. Berdsall and V. Foks [2], D. Van-Miter and D. Middlton [3-6]. Significantly earlier (in 1946), results, analogous to those from the theory of statistical solutions, were received independently of those known at that time on the theory of V. A. Kotel'nikov [36].

Here we presented only the first of the basic works in this field, determining the trend of the development of the theory. In the future, there appeared

many works, devoted to various particular aspects of the theory of radar detection, some of which we shall mention in the course of this account.

3.2. Main Positions of the Theory of Statistical Solutions

Towards the end of the 1930's and beginning of the 1940's, in mathematical statistics there existed two independent trends, connected with optimum methods of acceptance of solutions on the basis of random experiment: the theory of estimation and theory of check of statistical hypotheses. These trends were united in the 1940's by A. Wald [37, 38] into the general theory statistical solutions. In frames of this general theory were introduced some new conceptions, the use of which turned out to be very fruitful, and a number of new general results was received.

By the term "statistical solutions" we usually mean solutions, accepted on the basis of observation of some totality of random variables or the realization of a random process. Statistical solutions include, for example, the solution on the presence of a target or on the values of its coordinates, taken in radar on the basis of observed realization of a received signal. The theory of statistical solutions is occupied with the investigation, comparison and detecting of the best methods of acceptance of such solutions.

In this paragraph we will present the definitions of the main conceptions and will briefly enumerate the general results of the theory of statistical solutions. The expounded material refers, in an equal degree, to problems of radar detection and measurement of coordinates and will be used also in the chapters, devoted to measurement.

The totality y of observed meanings of random variables in mathematical statistics usually is called selection. This totality is described as a multi-dimensional (by the number of magnitudes, entered in y) distribution of probability. For simplicity, we shall consider that there exists a multi-dimensional probability

density p(y) (y may be considered as a multi-dimensional vector). In the future, we, basically, must deal with solutions, accepted on the basis of observation of realizations of random processes. These realizations, which are infinite-dimensional selections, can be described with the help of functionals of probability density (see Section 1.4). So that the obtained results are correct both for discrete selections and for realizations, we shall designate the multi-dimensional probability density and functional of density by one symbol p(y). Inasmuch as consideration of discrete and continuous cases is conducted completely analogously, such generalization will not lead to any incorrect conclusions.

In problems of statistical solution, the distributive law of p(y) is usually partially or completely unknown. Therefore, it is natural to consider p(y) as a conditional distribution, depending on a certain totality of unknown parameters s and to designate it by p(y/s). Parameter s may also be considered as a vector in a multi-dimensional space, which in the future we shall designate by S.

The accepted solution is determined by observed realization and the rule, in accordance with which it is taken. If the given solution is considered as element d of a certain great number D_o of possible solutions, then this rule may be considered as function d(y), depicting a great number of realizations Y on a great number of solutions D_o. Function d(y) is called a decisive function.

Let us consider the two simplest radar examples.

Example 1. Let, in magnitude of noise distorted received signal y(t), it be required to determine the magnitude of the reflecting surface of the observed target. In this case the space of the parameter of distribution represents the number-scale axis, on which are found the values of the reflecting surface. The great number of solutions also represents the number-scale axis. The reflecting surface of the target can be determined by the formula

$$\sigma_{\text{ц}} = C\frac{1}{T}\int_0^T y^2(t)\,dt, \tag{3.2.1}$$

where T is the time of observation;

y(t) —the received signal;

C — the proportionality factor, depending on distance to the target and parameters of radar.

Expression (3.2.1) also determines the decisive function.

<u>Example 2</u>. Let, on the basis of observations of a signal received during time T, it be required to accept the solution on the presence of the target, the parameters of which beforehand are given. In this case it is essential for the acceptance of the solution with an unknown parameter of distribution of probabilities of the received signal also to know the reflecting surface of target. If $s=\sigma_{\text{ц}}=0$, then the solution on the absence of the target will be correct. If $s=\sigma_{\text{ц}0}$, then the solution on the presence of the target will be correct. Thus, a great number of solutions of S consists of two points. A great number of solutions of D_0 in this case also consists of two elements: "target" (d_1), and "no target" d_0. The solution can be taken on the basis of comparison of signal power

$$E=\int_0^T y^2(t)\,dt$$

with certain threshold E_0. If y is such that $E>E_0$, then $d(y) = d_1$; if y such that $E<E_0$, then $d(y) = d_0$.

In most cases, the observed realization does not determine simply one of the possible situations s. There are possible errors in acceptance of solutions, leading to negative consequences, which are desirable to minimize by corresponding selection of a decisive function. If these consequences can be represented in quantitative form (in rubles, for example), then it is possible to introduce in consideration the function I(d, s), called the function of losses or risk.

At fixed s and d, the function I(d, s) is equal to the magnitude of losses, connected with the acceptance of solution d, when situation s takes place. Inasmuch as d is the function of observed realization y, the magnitude of I(d, s) at given s is random. A decisive rule can be characterized with the help of statistical characteristics of random variable I. Usually as such a characteristic is used the mathematical expectation

$$R(d/s)=\int_Y I[d(y), s]\, p(y/s)\, dy \qquad (3.2.2)$$

at given s, called conditional risk.

In applying the conception of conditional risk, it is possible to carry out a comparison, and sometimes a simple selection of one of the possible decisive rules. If for one of the rules, the conditional risk at all s is less than for the other (or others), then it is natural to give preference to this rule.

In most cases, the rule, minimizing conditional risk, is obtained at various s. In theories of statistical solutions are developed two methods of selection of the best decisive rule in such cases.

The first method assumes to be known an a priori distribution of possible situations s. For simplicity we shall consider that for that distribution there exists the probability density $p_0(s)$. Using the a priori distribution, we can compute the mean risk. The mean risk depends only on the accepted decisive rule and can be used for comparison of these rules.

$$R(d)=\int_S R(d/s)\, p_0(s)\, ds=\int_S\int_Y p_0(s)\, I[d(y)/s]\, p(y/s)\, ds\, dy. \qquad (3.2.3)$$

The decisive rule, for which the mean risk turns out to be the least, is called the Bayes solution relative to the considered a priori distribution $p_0(s)$, and corresponding mean risk—the Bayes risk. In the theories is proved the existence of the Bayes solution for arbitrary a priori distribution and limited non-negative function of losses.

In many practical problems, a priori distribution is unknown. Therefore, it is desirable to find methods of determining the optimum decisive functions, not depending on $p_0(s)$. Here, it is natural to demand that these methods ensure the best results in the worst situation.

The first of such methods is closely connected with the Bayes solutions. We shall introduce the least preferable a priori distribution, for which the Bayes risk is maximum. Application of the Bayes solution, corresponding to the least preferable distribution, is, to a certain degree, satisfactory, since it ensures minimum risk at the least favorable a priori distribution.

The other method consists of the use of the so-called minimax solutions. The minimax decisive rule is determined from the condition of the minimum of the highest value of conditional risk:

$$\min_{d} \max_{s} R(d/s). \qquad (3.2.4)$$

One of the fundamental results of the theory of solutions consists in the fact that under very general conditions, the minimax solution coincides with the Bayes' relative to the least preferable a priori distribution $q_0(s)$, where the conditional risk corresponding to this solution is constant for all s, with which $q_0(s) \neq 0$. This result considerably facilitates finding of the minimax solutions.

Until now we spoke of determined (non-random) decisive functions. It is possible to imagine such cases, when the accepted solution d is determined by the result of random experiment, selected in accordance with the observed realization of y. Here, the realization determines the distributive law of probabilities p(d/y), describing the utilized random experiment. Frequently the encountered form of such experiment is casting the die. In distinction from the determined decisive rules, the rules, using the random experiment, are called randomized. It is proven that at a finite number of possible situations s for any randomized decisive rule, there exists an equivalent determined rule. Equivalence in the

given case signifies the equality of conditional risks. For arbitrary sets of situations s under very general conditions is proven the existence of the determined rule, for which the conditional risk hardly exceeds the conditional risk for the given randomized rule (ε - equivalence). Thus, if the effectiveness of the decisive rule is characterized by magnitudes of conditional risks $R(d/s)$, then from this point of view, the use randomized rules does not give practically any gain.

In practice, we frequently encounter cases, when the dimensions of the observed realization beforehand are not fixed and are determined, proceeding from the required authenticity of the accepted solutions. A. Val'd [39] proposed to conduct, in such cases, an experiment in stages, and at every stage to take either one of the possible final solutions $d \in D_0$, or, if not one of these resolutions can be received with sufficient authenticity, take the solution on the continuation of the experiment. With such an approach, it is natural in the calculation of losses to consider the value of the experiment and to use, as a function of losses, the sum of the value of the experiment and losses, connected with erroneous solutions. For this case, all the above mentioned results of the theory of solutions, are distributed.

The methods of the theory of statistical solutions can be used in divers areas of human endeavor, connected with the statistical treatment of results of observations, and including radar, where obtained as a result of minimization of losses, the decisive rule can be interpreted as the optimum operations on the received signal and represented by corresponding optimum circuits of the receiving mechanisms.

Below will be presented more specific results of the theory of solutions, finding application in problems of detection of radar signals.

Before we cross to the account of these questions, it is useful to turn our attention to one aspect of the theory of solutions, connected with its

practical application and utility. If the function of losses is given, then the theory of solutions, in principle, allows us to find the decisive rule, ensuring the minimum of mean risk or minimax of conditional risk. In radar of such approach, we can remove the arbitrariness in questions of selection of methods of treatment of the received signal and produce a standard for estimating the quality of the technically realized methods of treatment. However, these decisive rules (methods of treatment) are optimum only with a fully determined function of losses, the conformity of which to the conditions of the considered problem is in an overwhelming majority cases very conditional.

For example, let us consider the problem of detection of a target, approaching the defense sector. How can we determine, in this case, the losses, connected with false alarm and omission of the target? How can we estimate quantitatively the consequence of panic among the population, caused by the false alarm, and dulling of vigilance of the maintenance personnel of the system of defense, occuring after the repulsion? Finally, how can we measure, in suitable units of value, the consequence of human sacrifice and destruction, caused by passage of the target to an object? Which approach—Bayes or minimax—should be used in the given problem? In what measure do we have the right to be oriented to minimization of average losses, allowing to receive an obvious gain in the multiple use of a decisive rule, when the question concerns the defense of a fully determined object maybe the only one of its kind? In any real problem there appears a great number of such questions; therefore, the function of losses is assigned usually very arbitrarily.

If it turned out that the form of the optimum decisive rule completely changes even with an insignificant change of the function of losses, and the magnitude of Bayes or minimax risk - with an insignificant change of the decisive rule, then optimization would completely lose its value. However, the numerous results of application of the theory of solutions, relating to various practical

problems, convince us more quickly of the existence of a reverse regularity. And namely, it turns out that with "reasonable" functions of losses, satisfying quite general conditions, obtained as a result of optimization of decisive functions are very close, just as the magnitude of the risk, corresponding to extremely various decisive rules. Sufficiently general results of such kind in the theory of solutions, unfortunately, are lacking.

In the theory of estimation and the theory of filtration, which are large branches of the general theory of statistical solutions, under certain limitations is proven the independence of the method of estimation from the form of function of losses, if the latter satisfies certain conditions of symmetry (see Chapter 6).

3.3 Double-Alternative Solutions

Let us consider the simplest problem of the theory of solutions, when on the basis of observation of realization y we should receive one of two alternatives, exclusive of one another, for example "target" and "no target". In this case any determined decisive rule represents a method of subdivision of a set of possible realizations y into two parts Y_0 and Y_1. If the observed realization y belongs to subset Y_0 ($y \in Y_0$), the solution d_0 is taken. If $y \in Y_1 = Y - Y_0$, solution d_1 is taken. The expression for average risk is recorded in this case in the form

$$R(d) = \int_{Y_0} \left[\int_S p_0(s)\, l(d_0, s)\, p(y/s)\, ds \right] dy + \int_{Y_1} \left[\int_S p_0(s)\, l(d_1, s)\, p(y/s)\, ds \right] dy. \qquad (3.3.1)$$

The sum of integrals in (3.3.1) will be least in the case when every realization y refers to that region of Y_0, Y_1, for which the integrand, in (3.3.1) upon integration by this region, is less. The following decisive rule follows from this: solution d_1 is taken, if for the observed realization y

$$\frac{\int_S p_0(s)\, l(d_0, s)\, p(y/s)\, ds}{\int_S p_0(s)\, l(d_1, s)\, p(y/s)\, ds} \geq 1; \qquad (3.3.2)$$

solution d_0 is taken, if an inverse inequality is carried out. The magnitudes in the numerator and denominator in the left part (3.3.2), may be considered as characteristic of a posteriori risk, connected with the acceptance of solution d_0 or d_1 during observation y. Actually, these magnitudes, with an accuracy up to the factor, not depending on s, coincide with the result of averaging I(d, s) by a posteriori distribution p(s/y). Thus, the received decisive rule has an evident value: that solution is taken, for which the a posteriori risk, corresponding to the realization y, is minimum.

Almost in all practical problems it is possible to indicate such subdivision of a set S of possible situations into subsets S_0 and S_1, which at $s \in S_0$, the solution d_0 is correct and there is no loss ($l(d_0, s) = 0$), and at $s \in S_1$, the solution d_1 is correct and $l(d_1, s) = 0$; whereupon $l(d_1, s) = l_0$, if $s \in S_0$, and $I(d_0, s) = I_0$, if $s \in S_1$. Thus, it can be done, for example, in the problem of detection, if by s we mean the reflecting surface of the target s_u. If $s_u = 0$ (no target), then the solution on the fact that there is none (d_0), is correct, and acceptance of the solution that there is a target (d_1), entails definite losses. If $s = s_u$, then acceptance of the solution d_1 does not lead to losses, and losses $I(d_0, s_u)$ (losses, connected with passage of target) may be considered not depending on s_u.

With the shown limitations from (3.3.2) we obtain

$$\Lambda(y) = \frac{P(y/S_1)}{P(y/S_0)} \geq \frac{l_1 P_0}{l_0 P_1} = c, \qquad (3.3.3)$$

where

$$P(y/S_i) = \frac{\int_{S_i} p_0(s)\, p(y/s)\, ds}{\int_{S_i} p_0(s)\, ds} \qquad (3.3.4)$$

represents the conditional probability of obtaining the realization y under the condition that $s \in S_i \ (i = 0;\ 1)$, and

$$P_i = \int_{S_i} p_a(s)\, ds. \tag{3.3.5}$$

Thus, in the considered case, the solution is taken on the basis of comparison with the threshold of the relation of conditional probabilities $\Lambda(y)$, called the relation of verisimilitude. The magnitude of threshold depends on the magnitudes of risks I_1 and I_0 and a priori probabilities P_1, P_0; it is possible to consider it as the relation of expected magnitudes of risk for cases $s \in S_0$ and $s \in S_1$ upon acceptance of knowingly incorrect solutions.

Inequality (3.3.3) can be treated as a comparison with the threshold of the relation of a posteriori probabilities

$$P(S_{1,0}/y) = \frac{P_{1,0} P(y/S_{1,0})}{\int_{S} p_a(s)\, p(y/s)\, ds}. \tag{3.3.6}$$

The solution d_1 is taken in the case when the a posteriori probability of which is correct, is I_1/I_0 times more than probability of which is incorrect.

The operation, determined by inequality (3.3.3), can obviously, be replaced by the comparison with the corresponding threshold of any monotonic function of the relation of verisimilitude. Usually as such a function is used a logarithm.

In problems of detection, a set S_0, for which the solution is correct that there is no target, usually consists of one element (parameters of noise or interferences are given), and set S_1 can contain a large number of elements (parameters of detected target can be various). Here,

$$\Lambda(y) = \int_{S_1} \Lambda(y,\ s)\, p_1(s)\, ds, \tag{3.3.7}$$

where $\Lambda(y,\ s) = \frac{p(y/s)}{p(y/s_0)}$ is the relation of verisimilitude for the case of a singular possible totality s of parameters of the target,

$p_1(s) = p_a(s)/1-p_0$ is the a priori distribution of parameters.

Thus, the relation of verisimilitude for detection of the target, the parameters of which are included in a certain region S_1, is obtained by avaraging the relations of verisimilitude for various specific values of $\underline{s}$ of parameters by all possible values of $s \in S_1$.

With these assumptions on the function of losses, the mean risk is determined by the formula

$$R(d) = l_0 \iint\limits_{Y_0 S_1} p_0(s)\, p(y/s)\, dyds + \\ + l_1 \iint\limits_{Y_1 S_0} p_0(s)\, p(y/s)\, dyds = P_1 \beta l_0 + P_0 F l_1, \qquad (3.3.8)$$

where β and F are probabilities of errors at $s \in S_1$ and $s \in S_0$ accordingly. In the theory of detection these probabilities are usually called probabilities of passage of the target (β) and false alarm (F).

For determination of the magnitude of risk it is necessary to calculate the probabilities β and F, corresponding to the considered decisive rule, and to substitute them in (3.3.8). Instead of the probability of passage, we frequently consider the probability of correct detection $D = 1 - \beta$.

In problems, for which the magnitudes of losses can be given only very conditionally, and a priori probabilities accurately are not known, it is justified to use, as characteristics of the decisive rule, not the magnitude of average risk, but the direct probability of errors of β and F and to find an optimum decisive rule, proceeding from the requirements, presented from any considerations to these probabilities. It is possible, in particular, to require minimum β at given F. Such a criterion of optimumness of the decisive rule is known under name of the criterion of Neumann and Pearson and is widely used in problems of double-alternative solutions. It is possible to show that the optimum decisive rule, corresponding to this criterion, also reduces to the comparison with the threshold of the relation of verisimilitude, whereby the magnitude of the threshold is determined by the given probability F.

In order to be convinced of this, it is sufficient to prove that any other rule gives a large probability of β besides F. Let there be sets of realizations Y_0 and Y'_0, such that at $y \in Y_0$ in accordance with the optimum rule is taken the solution d_0, and at $y \in Y'_0$, the same solution d_0 is taken in accordance with the considered nonoptimal rule. We shall designate by $Y_0 Y'_0$ the coinciding part of these sets, and by $Y_0 - Y_0 Y'_0$ and $Y'_0 - Y_0 Y'_0$ - their noncoincident parts. Then, considering that to set Y_0 belong all y, for wnich $P(y/S_1) \geq c P(y/S_0)$, we obtain

$$\beta' = \int_{Y'_0} P(y/S_1)\,dy = \int_{Y'_0 Y_0} P(y/S_1)\,dy + \int_{Y_0 - Y_0 Y'_0} P(y/S_1)\,dy + \int_{Y'_0 - Y_0 Y'_0} P(y/S_1)\,dy - \int_{Y_0 - Y_0 Y'_0} P(y/S_1)\,dy \geqslant \beta_0 + c\left[\int_{Y'_0 - Y_0 Y'_0} P(y/S_0)\,dy - \int_{Y_0 - Y_0 Y'_0} P(y/S_0)\,dy\right]. \tag{3.3.9}$$

By adding and subtracting in the right part of inequality $c \int_{Y_0 Y'_0} P(y/S_0)\,dy$, it is easy to check that the factor at c in the right part is equal to zero and, consequently, the probability β for the nonoptimal decisive ruled is greater, than for optimum.

In order to compare various decisive rules, not resorting to average risk, it is convenient to use the dependence β(F) or D(F) = 1 - β(F). Of the two decisive rules, the best, obviously, should be considered that, for which at the same F, the probability of D is greater.

In problems of detection, the dependence of probability of correct detection on the probability of false alarm D(F) is usually called the characteristic of detection. For the characteristic of detection, corresponding to the comparison of the relation of verisimilitude with the threshold, we establish a number of very interesting properties, not specifying the statistical properties of the observed signal.

If $p_{\text{сш}}(\Lambda)$ and $p_{\text{ш}}(\Lambda)$ are distributive laws for the relation of verisimilitude Λ in the presence and absence of a signal from the target (in the general case, of double-alternative solution at $s \in S_1$ and $s \in S_0$) accordingly, then for D and F we have

$$D = \int_{c}^{\infty} p_{\text{сш}}(\Lambda)\,d\Lambda,\quad F = \int_{c}^{\infty} p_{\text{ш}}(\Lambda)\,d\Lambda. \tag{3.3.10}$$

For explanation of the properties of the dependence D(F), let us consider the connection between $p_{\text{сш}}(\Lambda)$ and $p_{\text{ш}}(\Lambda)$. The characteristic function, corresponding to $p_{\text{ш}}(\Lambda)$, is determined by the expression

$$\psi_{\text{ш}}(\eta) = \overline{e^{i\eta\Lambda(y)}} = \int_{Y} e^{i\eta\Lambda(y)} P(y/S_0)\,dy = \\ = \int_{0}^{\infty} e^{i\eta\Lambda} p_{\text{ш}}(\Lambda)\,d\Lambda.$$

By differentiating $\psi_{\text{ш}}(\eta)$ by η and changing integration by y and differentiation by places, it is easy to check that

$$\frac{1}{i}\psi'_{\text{ш}}(\eta) = \int_{0}^{\infty} \Lambda p_{\text{ш}}(\Lambda)\,e^{i\eta\Lambda}\,d\Lambda = \int_{Y} e^{i\eta\Lambda(y)} P(y/S_1)\,dy = \\ = \int_{0}^{\infty} p_{\text{сш}}(\Lambda)\,e^{i\eta\Lambda}\,d\Lambda = \psi_{\text{сш}}(\eta),$$

whence

$$p_{\text{сш}}(\Lambda) = \Lambda p_{\text{ш}}(\Lambda). \tag{3.3.11}$$

The received relationship has a deep meaning and can be based purely on qualitative reasonings. Above it was proven that the optimum method of acceptance of the solution in the case of two alternatives is the comparison with the threshold of the relation of verisimilitude. Let us assume that we constructed an instrument, forming the relation of verisimilitude for observed realization y, and now want to select a method of acceptance of the solution. Here, in our reasonings, as the observed realization there should appear the relation of

verisimilitude and, in order to decrease the probability of errors, we should take the solution, by comparing with the threshold the relation of verisimilitude for this realization, i.e.,

$$\Lambda_2 = \frac{p_{cш}(\Lambda)}{p_{ш}(\Lambda)}.$$

Such a procedure may be repeated arbitrarily as much as desired, and considered consecutively $\Lambda_2, \ldots, \Lambda_n, \ldots$, where by the magnitude of the threshold, with which are compared the relations of verisimilitude of various orders, remains constant, since it depends only either on the values of the errors and a priori probabilities of the considered alternatives (in the case of minimization of average risk), or on the permissable probability F (for the criterion of Neumann and Pearson). All these decisive rules are optimum and should be equivalent. It follows from this that $\Lambda_2 = \Lambda_3 = \ldots = \Lambda$, i.e., (3.3.11)

By using (3.3.11), it is possible to establish a number of interesting properties of the characteristics of D(F), corresponding to the optimum decisive rule. For the derivative of $\frac{dD}{dF}$ from (3.3.10), (3.3.11) we receive

$$\frac{dD}{dF} = \frac{\frac{dD}{dc}}{\frac{dF}{dc}} = \frac{p_{cш}(c)}{p_{ш}(c)} = c(F). \qquad (3.3.12)$$

The magnitude of threshold, obviously, monotonically diminishes with the increase of F. Thus, the derivative of $\frac{dD}{dF}$ monotonically diminishes with the increase of F. By integrating (3.3.12), we obtain

$$D(F) = \int_0^F c(F)\,dF. \qquad (3.3.13)$$

In a whole number of cases, the use of these general relationships considerably simplifies the calculation of characteristics of detection. In the following paragraph we shall apply the received results for the solution of one practically important problem.

3.4. Comparison of "Detection in Interval" with "Detection by Points"

As was shown in Section 3.3 [see expression (3.3.7)], if the parameters of the targets are unknown and can take any values in a certain region S_1, then the relation of verisimilitude is obtained by averaging the relations of verisimilitude, calculated for various points of region S_1, by the whole region, taking into account the a priori distribution p_a (s). Usually, it is comparatively easily to construct a multichannel circuit, in each channel of which will be formed a voltage, proportional to the logarithm of the relation of verisimilitude Λ (y, s_i), where s_i is the totality of parameters of the target (distance, speed, angle), to which the given channel is tuned.

In order to receive an averaged relation of verisimilitude, it is necessary to pass the output voltage of each channel through a mechanism with an exponential characteristic and to sum in weight p_a (s). The realization of these operations is connected with known technical difficulties and in practice they usually prefer to compare with threshold log $\Lambda(y, s_i)$ in each channel, which is equivalent to the comparison with threshold $\Lambda(y, s_i)$. Such a method of acceptance of the solution is significantly simpler, than that connected with the averaging of $\Lambda(y/s)$ by s, and at the same time allows simultaneously with detection to estimate approximately the parameters of the detected target.

In connection with this, it is interesting to know about the comparison of optimum procedure of "detection in interval" with the method of "detection by points". This question already was considered for a particular case of exponential distribution of the logarithm of the relation of verisimilitude in [illegible], where the equivalence of these methods is proven at small F. Below is given a very lax basis of this result for the more general case.

We shall use the following idealizations. We shall consider that a priori are possible only of n values of parameters ($s_1, \ldots, s_n$), whereupon all these

values are equiprobable. We shall assume also that these values are distributed from each other sufficiently far, so that the corresponding relations of verisimilitude may be considered statistically independent, and that for all $\Lambda(y, s_i)$ the distributive law is identical.

With these conditions, the problem is formulated in the following manner: there is a sum of independent random variables $Z=\frac{1}{n}\sum_{i=1}^{n}\Lambda_i$, whereby either all Λ are described by the distributive law $p_{ш}(\Lambda)$, or n - 1 of them is subordinated to the law $p_{ш}(\Lambda)$, and one, to the law $p_{сш}(\Lambda)$. Distributive laws $p_{ш}(\Lambda)$ and $p_{сш}(\Lambda)$ are connected by the relationship (3.3.11): $p_{сш}(\Lambda)=\Lambda p_{ш}(\Lambda)$. It is required to establish a connection between probabilities of exceeding the threshold of sum Z and exceeding the corresponding threshold by at least one of the components in the first and in the second the above-indicated cases.

Let us consider at first the dependence of D(F) for the case, when with the threshold is compared the sum Z. Inasmuch as Z represents the averaged relation of verisimilitude, for distributive laws of this magnitude in the presence and absence of a target, the relationship (3.3.11) is correct, and dependence D (F) can be calculated by the formula (3.3.13). The corresponding magnitude of threshold we shall designate by C_n (F).

We shall now express C_n (F) by c_1 (f), considering $c_1(F) \gg 1$, which corresponds to sufficiently small F. The probability of $F_n(C)$ is such, that $nZ>c$, in the absence of a target it is possible to write in the form

$$F_n(c)=\int_c^{\infty}dx\int_0^x p_{ш}(z)\,p_{n-1}(x-z)\,dz,$$

where $P_{n-1}(x)$ is the convolution of n - 1 distributions $p_{ш}(z)$. By changing the order of integration and considering that

$$\int_c^{\infty}p_{n-1}(z)\,dz=F_{n-1}(c),$$

we obtain

$$F_n(c)=F_1(c)+\int_0^c p_{ш}(z)\,F_{n-1}(c-z)\,dz. \qquad (3.4.1)$$

Converting both parts of this equation by Fourier series, an equation can be found for

$$f_n(\lambda)=\int_0^\infty F_n(c)\,e^{i\lambda c}\,dc, \qquad (3.4.2)$$

from which it is simple to see that

$$f_n(\lambda)=f_1(\lambda)\frac{\psi_{\text{ш}}^n(\lambda)-1}{\psi_{\text{ш}}(\lambda)-1}. \qquad (3.4.3)$$

The probability of F_n (C) is determined from (3.4.3) by inverse Fourier conversion. It is known that with large arguments, corresponding to small F_n, the behavior of the function is determined by the behavior of its spectrum at small λ.

Here, by distributing into a series the factor at $f_1(\lambda)$ in(3.4.3) by powers of λ and considering that $\psi_{\text{ш}}(0)=1$ and $\psi'_{\text{ш}}(0)=i\psi_{\text{сш}}(0)=i$, after inverse conversion, we obtain

$$F_n(c)=nF_1(c)+\frac{n(n-1)}{2}p_{\text{ш}}(c)+\dots \qquad (3.4.4)$$

Decomposition (3.4.4) should be considered as an asymptotic estimate of F_n (c), correct under certain conditions for sufficiently large c. The necessary condition of application of this estimate is the equality

$$\lim_{c\to\infty}\frac{p_{\text{ш}}(c)}{F_1(c)}=0.$$

At

$$\frac{n-1}{2}p_{\text{ш}}(c)\ll F_1(c) \qquad (3.4.5)$$

it is possible to be limited in (3.4.4) to the first member and to determine the magnitude of threshold c, corresponding to the given probability F_n. Considering the presence of factor $\frac{1}{n}$ before the sum in the expression for Z and and substituting

$$c_n(F)\approx\frac{1}{n}c_1\left(\frac{F}{n}\right)$$

in (3.3.13), we obtain

$$D_n(F)=\int_0^F c_n(F)\,dF \approx D_1\left(\frac{F}{n}\right). \tag{3.4.6}$$

In the case of detection by points, by considering the independence of values of $\Lambda(y, s_i)$ in separate points, at $nF_1 \ll 1$, we have

$$F_n=1-(1-F_1)^n \approx nF_1 \tag{3.4.7}$$

$$D=1-(1-D_1)(1-F_1)^{n-1} \approx D_1(F)+\beta F_1(n-1). \tag{3.4.8}$$

Taking into account (3.4.7), the expression (3.4.8) may be written in the form

$$D_n(F_n) \approx D_1\left(\frac{F_n}{n}\right), \tag{3.4.9}$$

coinciding with (3.4.6). Thus, under imposed limitations, the considered methods of detection of the target are approximately equivalent. This result justifies the utilized methods of construction of systems of detection in the form of a totality of separate channels, tuned to various values of parameters of the target and filling with sufficient density the a priori interval of change of these parameters. The approximate equivalence of "detection in interval" and "detection by points" allows at theoretical consideration of a number of problems of detection to limited to the synthesis of optimum systems of detection at a point, using for the calculation of characteristics of "detection in interval" the formula (3.4.6). Here, it is necessary to consider condition (3.4.5), under which the shown equivalence takes place.

In the future we frequently will come in contact with the case, when the magnitude, connected with the logarithm of the relation of verisimilitude Λ by linear relationship, is distributed according to the law of chi-square with 21 degrees of freedom. Here, the dependence F_1 (c) can be written in the form

$$F_1(c)=\int_{a\ln bc}^{\infty}\frac{x^{l-1}}{(l-1)!}e^{-x}dx=\sum_{k=0}^{l-1}\frac{(a\ln bc)^k}{k!}e^{-a\ln bc} \tag{3.4.10}$$

where a and b are proportionality factors, depending on the signal-to-noise or signal-to-interference ratio. The distribution density $p_{m}(c)$ is obtained from (3.4.10) by differentiation by c. For relation $\frac{F_1(c)}{p_{m}(c)}$ we have

$$\frac{F_1(c)}{p_{m}(c)}=\frac{c}{a}\sum_{k=0}^{l-1}\frac{(l-1)!}{(l-k-1)!}\frac{1}{(a\ln bc)^k}, \tag{3.4.11}$$

from which it is clear that the considered relation tends to infinity upon tending c to infinity and at any n it is possible to take F so small, that condition (3.4.5) is fulfilled. Consideration of specific problems shows that to fulfill (3.4.5), it is sufficient that the general probability of false alarm for n channels is 5-10 times less than the probability of passage of the target. Thus, in the case of l = 1, the corresponding detection of a slowly fluctuating signal (see Chapter 4) in noises at signal-to noise ratio q_0, we have

$$\frac{F_1(c)}{p_{m}(c)}\,\frac{2}{n-1}=\frac{cq_0}{1+q_0}\cdot\frac{2}{n-1}\approx\frac{2\beta}{(n-1)F_1}\approx\frac{2\beta}{F_n}.$$

Given as the permissible, the probabilities of passage usually not less than 10^{-3}, when the probability of false alarm are usually several orders less, so that in real problems of detection, condition (3.4.5) is fulfilled.

3.5. Double-Alternative Solutions with Use of Sequential Analysis

In accordance with the general principle of continuation of the experiment, when not one of the final solutions can be received with sufficient authenticity, A. Wald proposed the following procedure of double-alternative solutions. At each moment of time t (at each stage of the experiment during discrete observations) the relation of verisimilitude, composed for the all observed realization $y(\tau)\,(0<\tau<t)$, is compared with two thresholds A and B. If $\Lambda_t(y)\geqslant A$, then solution d_1 is taken; if $\Lambda_t(y)\leqslant B$, d_0 is taken; if $B<\Lambda_t(y)<A$, the tests are continued. A. Wald developed in detail the procedure of acceptance of a solution

and calculated the corresponding characteristics for discrete independent observations [39]. Under certain conditions, these results can be wide-spread in the case of continuous time. Below we shall expound the results of sequential analysis in reference to this case.

We shall find, first of all, the connection between probabilities of errors of solution F (take d_1 instead of d_0) and β (take d_0 instead of d_1) and thresholds A and B. Region Y_1 of realizations y, the observation of which terminates in the acceptance of d_1, is determined by the equality

$$P_{t'}(y/s_1) = AP_{t'}(y/s_0),$$

where t' is the moment of first output of $\Lambda_t(y)$ beyond the limits of interval (B, A).

By integrating both parts of this equality by all y, for which it is fulfilled, we have

$$1-\beta = AF,$$

whence

$$A = \frac{1-\beta}{F}. \qquad (3.5.1)$$

Analogously, region Y_0 of acceptance of d_0 is determined by the equality

$$P_{t'}(y/s_1) = BP_{t'}(y/s_0).$$

Integrating both parts of the equality by all $y \in Y_0$, we obtain

$$\beta = B(1-F)$$

and

$$B = \frac{\beta}{1-F}. \qquad (3.5.2)$$

It is interesting to note that in the case of use of sequential analysis, F and β are not connected together and can be given independently. Besides these characteristics, in the use of sequential analysis an important role is played by the duration of the test. The most accessible for the determination of the statistical characteristics of duration t' which is random, is the mathematical

expectation $\bar{t}'$. For calculation of t', we introduce $L(t)=\log\Lambda_t(y)$ and shall assume that

$$L(t)=\int_0^t l_t(y)\,dt,$$

$$\overline{L(t)}=l_0 t, \tag{3.5.3}$$

where l_0 does not depend on t. We shall now take the interval of time T so great, that is is possible to affirm quite reliably that $t'<T$. Then on the strength of 3.5.3)

$$\overline{L(T)}=l_0 T=\overline{\int_0^{t'} l_t(y)\,dt}+\overline{\int_{t'}^{T} l_t(y)\,dt}=\overline{L(t')}+\overline{\int_{t'}^{T} l_t(y)\,dt}. \tag{3.5.4}$$

Function L (t'), on the strength of determination of t' takes one of two values, a = log A and b = log B. The second component in (3.5.4) is the mean value of the integral from the random function, one of the limits of which t' is random, where t' and $l_t(y)\,(t'<t<T)$, in general, are not independent. However, if T - t' is great as compared to the time of correlation of the process l_t (y), it is possible to consider

$$\overline{\int_{t'}^{T} l_t(y)\,dt}\approx l_0(T-\bar{t}'). \tag{3.5.5}$$

Substitution of (3.5.5) into (3.5.4) gives

$$\bar{t}'\approx\frac{\overline{L(t')}}{l_0}. \tag{3.5.6}$$

At $s\in S_0$

$$\bar{t}'\approx\frac{F\log A+(1-F)\log B}{l'_0}=\frac{f(F,\beta)}{l'_0}, \tag{3.5.7}$$

where $l'_0 = l_0$ at $s\in S_0$.

Analogously, at $s\in S_1$

$$\bar{t}'\approx\frac{(1-\beta)\log A+\beta\log B}{l''_0}=\frac{-f(\beta,F)}{l''_0}. \tag{3.5.8}$$

In these formulas

$$f(F,\beta)=F\log\frac{(1-\beta)(1-F)}{\beta F}+\log\frac{\beta}{1-F}. \tag{3.5.9}$$

If the losses, connected with the continuation of the experiment, are directly proportional to its duration, then, by using (3.5.7) and (3.5.8), there can be obtained an expression of average risk, depending on F and β, and the risk can be minimized to the corresponding selection of these probabilities.

A. Wald and J. Wolfowicz [41] showed that the considered decisive rule ensures a minimum of average duration of the experiment at given F and β, and thereby proved the optimumness of this rule.

Questions of application of sequential analysis to problems of detection of signals are contained in a number of works in the USSR and abroad [42--47]; however, on the ways of its practical use there still are principal difficulties. Some of them we shall consider here.

As already was noted, the duration t' of observation with the use of sequential analysis is random. The losses, connected with the continuation of the experiment, usually can be considered proportional to t' only on a certain finite section. Therefore, for a detailed explanation of the advantages of sequential analysis it is necessary to know the distributive law $p(t')$, the calculation of which for the majority of practical cases is an insuperable problem in the given stage of development of the theory. Even the calculation of mean value of t' is done only under very particular conditions (3.5.5), not carried out for a whole number of practically important cases (detection in interval, slowly fluctuating signal, etc.).

Certainly, the distributive law can be determined experimentally; however, in some problems, such a determination is not satisfactory. One of the most important of these problems is the problem of multichannel detection, when, before turning the antenna to another direction, it is necessary to take a solution on the presence of a target in one of the channels, untuned, for example, by distance, or on the absence of a target in all channels. The use of the classical procedure of sequential analysis with two constant thresholds in each channel

leads, in this case, to essential losses in time, since the transition to a new angular direction is possible only after in each channel there occurs a transition to one of the thresholds.

Thus, the full time of observation is determined in the case by the magnitude max t'_i, where $t'_i (i = 1, 2, \ldots, n)$ are the times of the observations, corresponding to the various channels. Obviously, max t'_i can many times exceed $\tilde{t}'$. In connection with this, in [47] was proposed the use of variable thresholds A and B; however, the calculation of optimum functions A(t) and B(t) is hampered by the absence of a common method of calculation of probability p(t'), which in this case should be known for arbitrarily variable thresholds.

The calculation of the optimum law of change of A(t) and B(t) is conducted in [47] by direct method (in steps) for the case of double storage of statistically independent pulse signals in the receiving mechanism. The gain in the threshold signal in the use of such a procedure of detection, instead of comparison with the threshold of the relation of verisimilitude, calculated in fixed time, constitutes 3--4 db at $n = 100$, $F \sim 10^{-10}$, $\beta = 0.1$. These results may be considered only as especially preliminary, since the conditions of their obtainment greatly differ from actual conditions. On the whole, this problem, which is of great theoretical and practical interest, still awaits its solution.

3.6. Multi-Alternative Solution. Detection of Target with Simultaneous Estimate of Its Parameters

We shall now complicate the problem, considered in Section 3.3, considering a number of mutually exclusive alternatives to be arbitrary $(d_0, d_1, \ldots, d_n)$. Any determined decisive rule, leading, at any y, to the acceptance of one of the solutions $d_0, \ldots, d_n$, reduces in this case to subdivision of a set of possible realizations Y into such subsets $Y_0, \ldots, Y_n$, that at $y \in Y_i : d_i$ $(i = 0.1, \ldots, n)$ is taken. The expression for average risk, corresponding to such a rule, has

the form

$$R(d)=\sum_{i=0}^{n}\int_{Y_i}dy\int_{S}p_0(s)\,l(d_i,\ s)\,p(y|s)\,ds. \qquad (3.6.1)$$

So that the magnitude of risk R(d) is minimum, it is necessary for every realization y to have such set Y_i, for which the integrand

$$R_i(d_i/y)=\int_{S}p_0(s)\,l(d_i,\ s)\,p(y/s)\,ds \qquad (3.6.2)$$

is minimum. It follows from this that the region Y_i is determined by the following inequalities:

$$\int_{S}p_0(s)\,l(d_i,\ s)\,p(y/s)\,ds<$$

$$<\int_{S}p_0(s)\,l(d_j,\ s)\,p(y/s)\,ds \qquad (j\neq i). \qquad (3.6.3)$$

The meaning of the received decisive rule is obvious: the solution is always taken, which is connected, at given y with the least risk. If at any y, d_i, d_k in (3.6.3) is attained an equality, then such y can be arbitrarily referred to one of sets Y_i, Y_k without a change of the magnitude average risk.

The necessity of selection of one of the many alternatives in problems of detection appears in those cases, when, simultaneously with the acceptance of the solution on the presence of a target, it is necessary to accept definite solutions relative to the properties of the detected target (for example, to determine the distance to the target or number of targets in a group).

Let us consider one of the most frequently encountered problems of such kind. Let a target appear in one of the points of the interval of change of distance (or speed) $\Delta\lambda$. It is required to detect the target and indicate the value of λ with an accuracy, sufficient for securing the target by the system of autotracking.

We shall consider that for this it is required to determine, to which n of nonoverlapping intervals $\Delta\lambda_i$ (their width $\delta\lambda=\frac{\Delta\lambda}{n}$ is equal to the width of the discrimination response of the system of tracking) belongs the true value of

parameter λ. There are n+1 possible solutions: there is no target (d_0), there is a target and $\lambda\in\delta\lambda_1\,(d_1),\ldots$, there is a target and $\lambda\in\delta\lambda_n\,(d_n)$.

We shall introduce three possible values of the function of losses: $l(d_i, s_0)=l_F$ ($i = 1,\ldots, n$) (losses, connected with false alarm), $l(d_0, S_i) = l_\beta$ ($i = 1,\ldots, n$) (losses, connected with passage of target), $l(d_i, S_k) = l_\text{и}$ ($i, k = 1,\ldots, n$) (losses, connected with incorrect indication of interval $\delta\lambda_i$, in which the target is located). Inequalities (3.6.3) thereby take the following form:

at $i,\ j\neq 0$

$$\int\limits_{\Delta\lambda-\delta\lambda_i} p_0(\lambda)\,p(y/\lambda)\,d\lambda<\int\limits_{\Delta\lambda-\delta\lambda_j} p_0(\lambda)\,p(y/\lambda)\,d\lambda, \tag{3.6.4}$$

At $j=0$

$$\begin{gathered} l_\text{и}\int\limits_{\Delta\lambda-\delta_{\lambda i}} p_0(\lambda)\,p(y/\lambda)\,d\lambda+ \\ +l_F\,p(s_0)\,p(y/s_1)<l_\beta\int\limits_{\Delta\lambda} p_0(\lambda)\,p(y/\lambda)\,d\lambda, \end{gathered} \tag{3.6.5}$$

where $P(s_0)$ is the a priori probability of absence of target.

At $i = 0$, we obtain an inequality, the reverse of (3.6.5).

From (3.6.4), (3.6.5) it follows that the solution on the presence of a target in the interval $\delta\lambda_i$ should be taken upon fulfillment of inequalities

$$\int\limits_{\delta\lambda_i} p_0(\lambda)\,p(y/\lambda)\,d\lambda>\int\limits_{\delta\lambda_j} p_0(\lambda)\,p(y/\lambda)\,d\lambda \quad (j=1,\ldots,n;\ j\neq i), \tag{3.6.6}$$

$$\begin{gathered} l_\text{и}\int\limits_{\delta\lambda_i} p_0(\lambda)\,p(y/\lambda)\,d\lambda>l_F\,p(s_0)\,p(y/s_0)- \\ -(l_\beta-l_\text{и})\int\limits_{\Delta\lambda} p_0(\lambda)\,p(y/\lambda)\,d\lambda. \end{gathered} \tag{3.6.7}$$

Practically, for a number of problems, the detection of a target not in that interval $\delta\lambda_i$, in which it is in reality, is equivalent in its consequences to passage of the target. Therefore, it is possible to place $l_\text{и}=l_\beta$. In

this case, the decisive rule, determined by inequalities (3.6.6), (3.6.7), reduces to the comparison of the averaged relation of verisimilitude

$$\Lambda_i(y)=\int_{\Delta_i} p_a(\lambda)\Lambda(y, \lambda)\,d\lambda \quad (i=1;\ 2;\ldots;\ n) \tag{3.6.8}$$

with the threshold

$$c=\frac{p(s_0)\,l_F}{l_\beta[1-p(s_0)]}, \tag{3.6.9}$$

where $p_a(\lambda)=p_0(\lambda)/[1-p(s_0)]$ is the a priori distribution λ under the condition that there is a target.

The relations of verisimilitude, exceeding the threshold, are selected and are compared to each other. It is considered that the target in this case is in the interval $\delta\lambda_i$, for which $\Lambda_i(y)$ has the highest value. If $p_a(\lambda)$ on interval $\delta\lambda$ is approximately constant, which usually takes place, and $l_\beta \ll l_F\,(F \ll \beta)$, then, as it was shown in Section 3.4, detection in interval $\delta\lambda$ is approximately equivalent to the comparison with the threshold of the relation of verisimilitude in each point. The magnitude of this threshold should be selected in such a manner that the general probability of false alarm remains constant. In a large number of cases, the interval $\delta\lambda$ is small or approximately coincides with the width of the interval of solution by parameter λ.

The average risk for the considered problem is entered in the form

$$R(d)=l_\beta\sum_{i=1}^{n} P_a(\delta\lambda_i)\beta_i+l_F\,p(s_0)F. \tag{3.6.10}$$

where $P_a(\delta\lambda_i)$ is the a priori probability of the presence of a target in interval $\delta\lambda_i$;

β_i is the probability of not detecting this target in the given interval, when it is there.

Analogously, it is possible to introduce the probability of false alarm F_i in the ith interval. For the calculation of R (d), one can use the properties of the dependences $\beta_i(F_i)$, considered in Sections 3.3 and 3.4.

Used in the consideration, the subdivision of a priori interval $\Delta\lambda$ into intervals of finite duration $\delta\lambda$ is, for a large number of practical problems, not wholly a justified idealization. We, as if, consider that the tracking system, intended for automatic tracking of the detected target, can be tuned only to determined discrete values of parameter λ and the problem of the system of detection is the indication of those of the values, which are closest to being true. Such situations, undoubtedly, can take place; however, more frequently it is required to directly indicate the position of the detected target with accuracy, sufficient for lock-on by the tracking system or for the completion of any other problems. Here, on the basis of observed realization y is taken either the solution on the absence of a target, or the solution on the presence of a target with a determined value of parameter λ (or several parameters).

The problems of determination of unknown parameters of the distributive law, on the basis of observation of some realization, described by this law, in statistics are called problems of estimation. The theory of estimation, with necessary detail, will be presented in chapter 6, volume II. Here we will touch only certain questions of theories, having direct relation to problems of detection with simultaneous estimation of parameters of a target. Here, we shall consider the parameters to be unvariable during the time of detection (i.e., consider their change to be small as compared with the corresponding intervals of the solution).

Let us consider for simplicity, the case of one parameter. (All reasonings, conducted for that case, are directly generalized in the case of several parameters). We shall introduce the function of losses $\omega(\dot{\lambda}, \lambda)$, connected with the error of measurement, depending on true (λ) and estimated $(\dot{\lambda})$ values of the parameter. The problem of estimation coincides, obviously, with the problem of a multi-alternative solution with an infinite number of alternatives. In accordance with (3.6.3), the solution on the presence of a target with parameter

$\hat{\lambda}_1$ should be taken, if the following conditions are carried out:

$$\int_{\Delta\lambda} [w(\hat{\lambda}, \lambda) - w(\hat{\lambda}_1, \lambda)]\, p_0(\lambda)\, p(y/\lambda)\, d\lambda \geq 0 \text{ at } \hat{\lambda} \neq \hat{\lambda}_1. \tag{3.6.11}$$

$$\int_{\Delta\lambda} [l_{\jmath} - w(\hat{\lambda}, \lambda)]\, p_0(\lambda)\, p(y/\lambda)\, d\lambda > l_t\, p(s_0)\, p(y/s_0). \tag{3.6.12}$$

The procedure of acceptance of a solution, corresponding to these inequalities, consists in the following. For all $\hat{\lambda}$, we check the fulfillment of inequality (3.6.12). If this inequality is carried out at least with one value of $\hat{\lambda}$, then the target is considered to be detected. As the estimated value of the parameter of the detected target, we select that value $\hat{\lambda} = \hat{\lambda}_1$, with which is attained

$$\min_{\hat{\lambda}} \int_{\Delta\lambda} w(\hat{\lambda}, \lambda)\, p_0(\lambda)\, p(y/\lambda)\, d\lambda. \tag{3.6.13}$$

We shall introduce the function of losses of the form

$$w(\hat{\lambda}, \lambda) = l_{\jmath}\left[1 - \mathrm{rect}\left(\frac{\hat{\lambda} - \lambda}{\delta\lambda}\right)\right]. \tag{3.6.14}$$

where $\delta\lambda$ is the permissible error, and

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \leq \frac{1}{2}, \\ 0 & |x| > \frac{1}{2}. \end{cases} \tag{3.6.15}$$

We consider, thereby, that there are no losses, if the error $|\hat{\lambda} - \lambda|$ does not exceed the determined magnitude $\delta\lambda$ (for example, the half-width of the discrimination curve of the tracking system), and is equal to the losses upon passage, if $|\hat{\lambda} - \lambda| > \frac{\delta\lambda}{2}$. With the introduced function of losses, the solution on the presence of a target is taken on the basis of comparison of the verisimilitude relation

$$\Lambda_0(y, \hat{\lambda}) = \int_{\hat{\lambda} - \frac{\delta\lambda}{2}}^{\hat{\lambda} + \frac{\delta\lambda}{2}} p_0(\lambda)\, \Lambda(y, \lambda)\, d\lambda \tag{3.6.16}$$

with the threshold (3.6.9) at all $\hat{\lambda}$. If at least in one point $\Lambda_0(y, \hat{\lambda}) \geq c$, then the solution is taken that there is a target, and as the estimated value

of parameter $\hat{\lambda}_1$ is selected that value of $\hat{\lambda}$, at which $\Lambda_0(y, \hat{\lambda})$ is maximum. In accordance with the results of section 3.4 under certain conditions, the comparison with the threshold $\Lambda_0(y, \hat{\lambda})$ can, without losses be replaced by comparison with the threshold $c_0(\lambda)$ of the relation of verisimilitude $\Lambda(y, \lambda)$ in each point of λ. Selection of the maximum value of $\Lambda_0(y, \lambda)$ at sufficiently small $\delta\lambda$ also can be replaced by selection of the maximum of $p_0(\lambda)\Lambda(y, \lambda)$. Inasmuch as in an overwhelming majority of problems, the a priori distribution is unknown, it is expedient to make $p_0(\lambda)$ equal to the least preferable distribution, which usually turns out to be uniform. Here, the dependence of threshold $c_0(\lambda)$ on λ disappears.

Obtained as a result of these simplifications, the procedure of acceptance of a solution is close to that practically utilized. Usually, the a priori interval of change of parameter $\Delta\lambda$ is filled with sufficient density by channels, tuned to the fixed values of $\lambda_1, \ldots, \lambda_m$. The output of every channel is compared with the threshold, and to the target is added the value of parameter $\hat{\lambda}$ corresponding to the channel, in which exceeding of the threshold occured. If the width of interval $\delta\lambda$ is greater than the resolving power in parameter λ, then such a method of determination of $\hat{\lambda}$ is practically equivalent to the above considered, since operation in channels, detuned relative to the target signal by more than the magnitude of the interval of solution, occurs only due to noises and, consequently, with very low probability. The advantage of this method, besides its relative simplicity, is the fact that it can be, with sucess, used for simultaneous detection of several targets, occcurring in a priori interval $\Delta\lambda$.

If the radar set, in the composition of which is the considered system of lock-on, operates on the target designation data, then the presence of a target in a priori interval $\Delta\lambda$ is considered to be an established fact and it remains to estimate the magnitude of its parameter. The decisive rule in this case is determined by equation (3.6.13). For the function of losses (3.6.14)

at quite low $\Delta\lambda$, the estimate of $\hat{\lambda}$, determined from (3.6.13), practically coincides with the estimate according to the maximum of a posteriori probability

$$\max_{\lambda} \frac{p_a(\lambda)\, p(y/\lambda)}{\int\limits_{\Delta\lambda} p_a(\lambda) p(y/\lambda)\, d\lambda} \sim \max_{\lambda} p_a(\lambda)\, p(y/\lambda). \qquad (3.6.17)$$

If distribution $p_a(\lambda)$ is considered uniform, then $\hat{\lambda}$ is determined by the maximum of any monotonically increasing function of the relation of verisimilitude. In practice, usually in this case, the selection of $\hat{\lambda}$ is made on the basis of comparison $\Lambda(y, \lambda)$ with the threshold, the exceeding of which in the presence of some noises is probably low. Such a method can be used upon falling of several targets in the interval of target designation. Comparison of these methods, presenting significant interest, can be made only for certain particular cases (see Chapter 4).

Instead of the function of losses (3.6.14), others can be used, for example, quadratic $[w(\hat{\lambda}, \lambda) = (\hat{\lambda} - \lambda)^2]$. If the a priori distribution $P_a(\lambda)$ changes slowly as compared to function $p(y/\lambda)$ (function of verisimilitude), and the function of verisimilitude is symmetric relative to some $\hat{\lambda}$ within the limits of its main maximum, then the optimum estimates, corresponding to symmetric functions of losses $w(\hat{\lambda} - \lambda)$, approximately coincide with the estimate of the maximum of a posteriori probability (see Chapter 6 Table II).

3.7. Sufficient Statistics. Principle of Inverse Probability

In any problem of statistical solution, the form of the optimum decisive rule is determined by such factors as the character of the taken solutions, the utilized criterion of optimumness, the presence and authenticity of a priori information, etc. It would have been possible to expect that the change of any of these factors leads to an essential change of the decisive rule, to the change of conversions, which is subjected to the observed realization in the process of solution. However, the results of the preceding paragraphs show that this, at

least, is not always so. In all considered problems, the solution was taken on the basis of comparison of certain integrals from the function of verisimilitude p(y/s) or the relation of verisimilitude $\Lambda(y/s)$. Thus, the formation of a functi or relation of verisimilitude for the observed realization y enters all the decisive rules considered optimum from any point of view. This is the result of a more common regularity, discussed below.

The observed selection or realization, on the basis of which the solution is taken, along with the useful information, concerning the taken solution, contains unnecessary information. For example, the taken signal contains information on the level of noises of the receiving mechanism. In the process of treatment of a signal, preceding the acceptance of the solution (for example, in the formation of the relation of verisimilitude, the preceding comparison of this relation with thethreshold), the unnecessary information is partially eliminated. Here, if treatment of the signal is optimum, during all conversions of the signal, the useful information should be completely kept, so that the result of each conversion can be used instead of the initial realization upon acceptance of a solution without decrease of authenticity of this solution. The results of conversions of such kind is called sufficient statistics [48]. They say that the conversion $\gamma(y, S)$ preserves all useful information on $s \in S$, contained in y, if for any method of acceptance of solution on s on the basis of y there exists a method of acceptance of solution on s on the basis of $\gamma(y, S)$, leading to the same distributive law for the taken solutions at any s. Here, any risk, accessible with the use of y (at any function of losses), is also accessible with the use of $\gamma(y, S)$

From the determination it is obvious that sufficient statistics are, in particular, the very realization of y and the results of any one-to-one conversions of this realization (for example, amplification). However, these sufficient statistics contain, along with useful information also all unnecessary information

contained in y. The unnecessary information is eliminated only with nonreversible conversions, with which the same result is obtained for a certain totality of initial realizations. Among the nonreversible conversions there exists [48] such, with which the unecessary information is eliminated in a maximum degree, possible under the condition of preservation of useful information. The result of such conversion is called the minimum sufficient statistics. In the theory it is proven [48], that under quite general conditions of minimum sufficient statistics, is the totality of values of the relation of verisimilitude $\Lambda(y, s)$ at all s, and also, obviously, any one-to-one functions of this relation.

It follows from this that the receiving mechanism, on the output of which will be formed one of the minimum sufficient statistics (for example, log $\Lambda(y, s)$ for all a priori of possible s), is sufficient in the sense that the output data of such a receiver can be used instead of y upon acceptance of any solutions, concerning s, without a decrease of authenticity of these solutions. Any subsequent nonreversible conversions of these output data should be directly connected with the form of a priori distribution and the character of the taken solutions. The results of these conversions already will be sufficient statistics not for any solutions relative to s, but only for fully determined ones. Such a sufficient receiver should be considered optimum in those cases, when the character of the taken solutions can change in the process of operation of the radar set and the requirements for decisive rules cannot be formulated clearly enough. This, in particular, refers to that case, when solutions are taken by an operator, who is guided by vague, in general, considerations, not always adding to the quantitative calculation.

A sufficient receiver may be imagined in the form of a totality of channels, in each of which will be formed, for example, log $\Lambda(y, s)$ for one definite value of s. In certain problems, not all channels can be used, but only a determined part. Thus, in the construction of an optimum discriminator of a tracking system, it is sufficient to use two channels, detuned on the corresponding parameter.

Optimum systems of detection usually switch all channels in the a priori interval of parameters of the target. The continuum of channels for all s, as already was noted earilier, usually is replaced by a finite number of channels, detuned on the corresponding parameters so little, that $\Lambda(y, s)$ from channel to channel changes unessentially.

The described approach to optimization of radar receivers, with which the receiver is considered to be optimum, if on its output will be formed minimum sufficient statistics, was formulated in [49]. Significantly earlier in the works of F. Woodward and I. Davis [50, 8] was formulated the so-called principle of inverse probability, according to which, the output of the optimum receiver should be any monotonic function of a posteriori probability distribution for possible situations

$$p(s/y) = \frac{p_0(s)\, p(y/s)}{\int\limits_S p_0(s)\, p(y/s)\, ds}. \qquad (3.7.1)$$

Here, the authors originated from the fact that a posteriori distribution completely characterizes the uncertainty with respect to the real situation s, kept after reception of signal y, and, consequently, contains all information on s, contained in y. Formally, the principle of inverse probability is a particular case of a more common approach, connected with the conception of sufficient statistics [p (s/y) — sufficient statistics]; practically, however, the results, obtained with the help of both approaches, coincide.

3.8. Optimization of Space Scanning

A number of very important and complicated problems appears in connection with rational selection of a method of scanning space by radar.

Scanning can be carried out in distance and in speed in those cases, when the system of detection cannot be made multichannel by these parameters (as this is required, proceeding from the above well-developed theory) due to

technical causes. In distinction from distance and speed scanning, scanning at angles is, in most cases, necessary in principle, since the sector, in which the target is detected, usually considerably exceeds the width of the beam of the antenna.

A significant part of the problems, connecting with scanning cannot be solved in accurate formulation due to the absence of sufficiently effective mathematical methods. We shall be limited here the solution of some simpler problems under rather strong idealizing assumptions. The received general results will be partially specified in subsequent chapters.

During the analysis, we shall distinguish conditions of scanning space and searching. By scanning, we mean the conditions, characteristic for radar sets of remote detection, when the a priori information about number and position of targets is completely lacking. The problem of such radar is the solution of the question of the presence of a target in each element of resolving power, into which it is possible to conditionally divide all the inspected space. The conditions of searching we shall call the conditions, characteristic for radar sets, working on the target designation data. During searching it is possible to consider known the a priori probabilities for number and position of targets. The form of corresponding distributions is determined by the distribution of possible errors of target designation. Such separation is, certainly, very conditional and connected exclusively with considerations of convenience of solution of the problem and the exposition. As the result of further development of the theory, the problems, related to these conditions, possibly, can be united.

In scanning conditions, a problem of the radar set is target detection with a high probability for a rather short time at a quite low frequency of false alarms. The probability of correct detection D is a function of the time of detection T_0, frequency of false alarms / and signal-to-noise ratio (or signal-

to-interference) q

$$D=D(T_0, F, q). \qquad (3.8.1)$$

The form of the function depends on the method of scanning. Usually, scanning is produced with constant speed, the selection of which can be the first step on the way to optimization. Selection of speed of scanning determines the number of cycles $n_{\text{ц}}$, into which the detection time is divided. Joint processing of signals, taken in various cycles, requires usually a very large quantity of memory cells in the system of processing, since at contemporary speeds, the detected target, in time T_0 can be transferred to a very large number of intervals of solution in distance and speed. In order to be convinced of this, let us consider the simplest numerical example.

Let us consider only solution in distance; the interval of solution constitutes 50 m, radial speed of the target $v_{\text{ц}}$ = 100 — 1,000 m/sec, detection time T_0 = 5 sec. During the time T_0, the distance to the target can change by 500 — 5,000 m, i.e., by 10 — 100 intervals of solution. Inasmuch as the speed of the target beforehand is unknown, we are forced to anticipate processing of the signal, taken in each element of solution in distance in the first cycle, jointly with the 90 elements of solution in the last cycle and, consequently, have in the receiver 90 channels of such joint processing on each element of solution in distance.

In connection with the noted circumstance, it is expedient to make a comparison with the threshold of the result of processing of the signal in each cycle separately. Here, inasmuch as such a method differs from the optimum, it is reasonable to decrease the number of cycles, in order to lower the influence of nonoptimalness of processing.

However, here is another factor, acting in the opposite direction. Such factors are fluctuations of a signal reflected from the target. If signals from the target in various cycles are statistically independent, then the

probability of small values of reflecting surface in all cycles decreases with the increase of their number. Thereby, the influence of fluctuations on the magnitude of probability D decreases. The influence of these two factors stipulates the existence of optimum number of statistically independent cycles. More specifically, these questions will be considered in the next two chapters. Here we will need only the conclusion that the detection time should be subdivided only into statistically independent cycles (bear in mind the fluctuations of a signal in neighboring cycles), if, certainly, subdivision is not connected with any other considerations.

If, in subdivision into cycles, the solution on the presence of a target is taken with at least one exceeding of the threshold during time T_0, then the memory in the system of detection from cycle to cycle can be completely absent. In principle, memorization of the fact of exceeding of the threshold in the computer of the radar set does not amount to much. Therefore, in consideration, it is expedient also to include such decisive rules, with which the solution on the presence of a target is taken with the appearance of a series of exceedings of the threshold (at least k from $n_ц$), occurring in various elements of solution, if the trajectory, passing through these elements, is not improbable for the considered class of targets. Here, there arises the question about selection of number k at given $n_ц$

If fluctuations of a signal in neighboring cycles are independent, and the signal-to- interference ratio during time T_0 changes little, then the number of exceedings of threshold in n cycles is subordinated to binomial distributive law Here, for probabilities of false alarm and correct detection the following expressions take place:

$$F=\sum_{i=k}^{n}\binom{n}{i}p_{ш}^{i}(1-p_{ш})^{n-i};\quad D=\sum_{i=k}^{n}\binom{n}{i}p_{сш}^{i}(1-p_{сш})^{n-i}, \qquad (3.8.2)$$

where $p_{ш}$ and $p_{сш}$ are probabilities of exceeding of threshold in the absence

and in the presence of a signal from the target accordingly.

If $\frac{n}{2} p_{\text{ш}} \ll 1$ and $\frac{n}{2}(1-p_{\text{сш}}) \ll 1$, which frequently takes place, then (3.8.2) may be replaced by approximate relationships

$$F \approx \binom{n}{k} p_{\text{ш}}^{k}, \quad D \approx 1 - \binom{n}{k-1}(1-p_{\text{сш}})^{n-k+1}. \tag{3.8.2'}$$

Considering the dependence $\beta_1(p_{\text{ш}}) = 1 - p_{\text{сш}}(p_{\text{ш}})$ to be known, from (3.8.2') we obtain

$$\beta = 1 - D \approx \binom{n}{k-1}\left[\beta_1\left(\frac{\sqrt[k]{FT}}{\sqrt[k]{\binom{n}{k}}}\right)\right]^{n-k+1} \tag{3.8.3}$$

where T is the part of total time T_0, occurring on one element of solution.

By minimizing this expression by k, we can find the optimum number of exceedings of threshold (see Chapter 4).

In principle, it is possible to abandon the uniform scanning and change the speed of scanning in accordance with the results of observations, quickly examining the direction, where the a posteriori probability of the presence of a target is low, and delaying in the directions, suspicious in the sense of presence of targets. We arrive, thereby, at dynamic programming of scanning [11]. Upon detecting the optimum program of scanning various criteria of optimumness can be used: minimum average time of scanning, maximum probability of detection of each appearing target in a given time after its appearance, etc. In the solution of these problems, one should, in general, consider the change of the signal-to-noise ratio in the approach (or departure) of the target.

If the criterion is used of minimum average time of scanning with given probabilities of correct detection and false alarm, then for the class of problems, satisfying conditions (3.5.3), the optimum method includes, obviously, the use of sequential analysis. If the system of detection is multichannel, then, as was noted in Section 3.5, the procedure of sequential analysis should be modified. Effective methods of detecting of optimum means of scanning for

that case, as for other criteria of optimumness, at present not are developed. This problem can be simplified by limiting the considered methods of scanning.

In particular, it is possible to assume that scanning is produced in several stages, whereby in every stage are examined only those directions, in which there occured exceeding of threshold on the preceding stage [11], and the time of observation at each stage is constant. If the signals, taken from the target in various stages of scanning, are considered to be statistically independent, then for the probability of passage of the target at m-stage scanning, B_m (F, T), we have

$$B_m(F,T) = \beta_1(F_1,T_1) + (1-\beta_1)\beta_2(F_2,T_2) + \ldots + (1-\beta_1)\ldots \ldots(1-\beta_{m-1})\beta_m(F_m,T_m), \qquad (3.8.4)$$

where β_i (F_i, T_i) is the probability of passage of the target on the ith stage;

F_i is the probability of false alarm;

T_i is the time of observation on the ith stage.

The general probability of false alarm and average duration of scanning without targets are determined by relationships

$$F = F_1F_2\ldots F_m \approx \frac{l\bar{T}_0}{l} = l\bar{T}, \qquad (3.8.5)$$

$$T_0 = l\bar{T} = l(T_1 + F_1T_2 + \ldots + F_1F_2\ldots F_{m-1}T_m), \qquad (3.8.6)$$

where l is the number of elements of resolving power in angles in the sector of scanning. With k targets in the sector of scanning, the time of scanning is increased by k (T_1 + T_2 +...+ T_m). Inasmuch as $\frac{k}{l}$ is usually very low, the relative increase of $\bar{T}$ is insignificant and it is possible not to consider it. Here, the formulation of the problem is obtained more clearly and the solution is not connected with the a priori information on the presence and on the quantity of targets.

For an m-stage scanning, it is possible to formulate a number of very interesting problems on the minimum. It is possible, for example, to find

$F_1, F_2, \ldots, F_m$ and $T_1, T_2, \ldots, T_m$, ensuring the minimum $B_m(F, \bar{T})$, or to find minimum $\bar{T}$ at given F and B_m. The solution of these problems can be reduced to the solution of functional equations, the form of which is characteristic for problems of dynamic programming [51]. For example, the first of the formulated problems reduces to the equation

$$B_m(F, \bar{T}) = \min_{F_1 T_1}\Big\{\beta_1(F_1, T_1) + [1 - \beta_1(F_1, T_1)]B_{m-1}\left(\frac{F}{F_1}, \frac{\bar{T} - T_1}{F_1}\right)\Big\}, \qquad (3.8.7)$$

ensuing directly from (3.8.4) — (3.8.6). The meaning of this equation is clear from the following reasonings.

Let F_1 and T_1 be selected from any considerations. At given $\bar{T}$ and F, the average time of scanning and probability of false alarm for subsequent stages are determined by relationships [see (3.8.5), (3.8.6)]

$$\bar{T}' = \frac{\bar{T} - T_1}{F_1}, \quad F' = \frac{F}{F_1}.$$

So that the general probability of passage $B'_m(F, \bar{T})$ is minimum at given T_1 and F_1, it is necessary to select $T_2, \ldots, T_m$; $F_2, \ldots, F_m$ from the conditions of minimum

$$B_{m-1}(F', \bar{T}') = B_{m-1}\left(\frac{F}{F_1}, \frac{\bar{T} - T_1}{F_1}\right).$$

After this is done for any T_1 and F_1, for final minimization of $B_m(F, \bar{T})$ it remains to select T_1 and F_1 in such a manner that the expression in brackets in (3.8.7) is minimum.

In [51] under very general conditions is proven the existence and singularity of the solution of equations of the form (3.8.7), and also convergence of the method of successive approximations for these equations. This method is, probably, only, a rather general method of solution of equations of the considered form.

By assigning a specific form of dependence $\beta_1(F_1, T_1)$ and using equation (3.8.7), it is possible, in principle, to find an optimum method and to estimate the effectiveness of multistage scanning. For certain particular cases, these questions will be considered in subsequent chapters.

3.9. Optimization of Target Search

Let us now turn to the consideration of questions of target search by radar, operating on target designation data. An account of these questions will be made, partially being based on results, received by I. N. Kuznetsov.

Let, in moment t = 0, be issued a target designation on the presence and the position of a target. Inasmuch as the target designation on the postition of the target is given with an error, for lock-on by angle, distance or speed, in most cases it is necessary to search in the interval of target designation, the magnitude which is determined by the distributive law of errors. This distributive law represents a priori distribution of positions of a target for the problem of searching.

We shall introduce function $g(t)$, characterizing the losses, connected with the delay of lock-on in t sec. Delay t is, in general, a random variable and is described by a certain distributive law p(t/s), depending on target coordinates s and the method of searching. Every method of searching can be characterized by the magnitude of conditional risk at various values of target position data or by the magnitude of average risk, determined by the expression

$$R = \int_S ds \int_0^\infty p_0(s)\, g(t)\, p(t/s)\, dt. \qquad (3.9.1)$$

By minimizing average risk R in all possible methods of searching, satisfying certain conditions, it is possible, in principle, to find the optimum method of searching, corresponding to the considered function of losses and given conditions. The necessity of assignment of additional conditions is

is completely evident, since equality (3.9.1) does not consider in itself the possibilities of false lock-on. The form of these conditions can be very different. As in the case of scanning, it is possible to assign a permissible average frequency of false alarm or probability of at least one false alarm from the moment of the beginning of searching, to lock-on. In those cases, when false lock-on leads only to losses in time (for the duration of a certain time, until a breakdown occurs, a false target is accompanied), additional conditions can be formulated in the form of requirements in duration of reliable work (without breakdowns) after target lock-on. We can assign, for example, an average time of work by signal or the probability of work by signal for the duration of a given time. Let us divide the inspected sector into unit cells, the size of which is determined by the resolving power of the radar by the corresponding coordinates, and number the cells in order of examination. Let us consider that the search is made in jumps from cell to cell, and the duration of a jump will be disregarded. Let us assume, furthermore, that the signal from the target is taken only when the system of searching is in the same cell as the target. A model of searching, obtained as a result of these simplifications, accurately reflects the main lines of the considered problem and at the same time allows to avoid excessive complications, connected with the form of directional diagram of the antenna, form of the strobe and so forth. For this model, formula (3.9.1) can be copied in the form

$$R=\sum_{\mu} p_{\mu} \int_{0}^{\infty} g(t)\, p(t/\mu)\, dt, \qquad (3.9.2)$$

where p_{μ} is the a priori probability of the presence of a target in μth cell;

$p(t/\mu)\,dt$ is the probability of detection of a target in μth cell, in interval of time (t, t+dt).

Delay t represents the sum of delay times τ_j in cells, the examination of which preceded detection. If magnitudes τ_j (being, in general, random, for

example, in the use of sequential analysis) may be considered statistically independent, then the characteristic function is obtained by multiplication of characteristic functions τ, and averaging of the product by the number of cells, which due to the possible passages of the target, is also random.

For the considered model of searching, optimization can consist in the selection of the order of examination of cells and characteristics of decisive rules (time of delay, magnitude of thresholds, etc.), utilized in the acceptance of solutions on the presence of a target in each cell. In order to simplify the solution, we shall assume these decisive rules to be identical. Analysis of a more general case requires very laborious calculations. As showed the consideration of certain particular problems, the gain in the threshold signal, as compared with the case of equally justified cells, constitutes 15 — 20%.

Selection of parameters of the system of lock-on (time of delay in cell, probability of passage at given frequency of false alarm), of minimizing average risk, requires assignment of a specific form of characteristic of detection in a cell. Therefore, we shall postpone consideration of this problem to subsequent chapters and be limited here to the selection of order and examination of cells. The function of losses will be considered as a linear function of time. Here, average risk is proportional to average time of searching $\bar{t}$, to the minimization of which we will tend.

We shall start from the simplest case of cyclical searching, with which a whole a priori interval of target position data is examined in a pre-determined selected sequence. Such a method of searching is also used in an overwhelming majority cases in practice. Under the condition that the target, in μth cell, was missed in $\underline{l}$ cycles and detected in ($\underline{l}$ + 1) th cycle, the average time of searching $\bar{t}_l$ is determined by the formula

$$\overline{t_l(\mu)} = \bar{\tau}_c' + (\mu - 1)\bar{\tau} + l\bar{\tau}_0 + l(N-1)\bar{\tau}, \qquad (3.9.3)$$

where $\bar{\tau}_c'$ -- average delay time in cell with target (with signal), when target

is detected;

$\bar{\tau}_c$ -- the same average time, when the target is missed;

$\bar{\tau}$ -- average delay time in cell, in which target is absent;

N -- number of cells in a cycle.

The probability of detecting a target in the (l + 1) th cycle is equal to $(1-\beta)\beta^l$ (β is the probability of passage); therefore, as a result of averaging (3.9.3) by l, we obtain

$$\bar{t}(\mu) = (\bar{\tau}'_c - \bar{\tau}) + \mu\bar{\tau} + \frac{\beta}{1-\beta}(\bar{\tau}_c + (N-1)\bar{\tau}), \qquad (3.9.4)$$

whence

$$\bar{t} = \sum_{\mu} p_{\mu}\bar{t}(\mu) = (\bar{\tau}'_c - \bar{\tau}_c) + \bar{\tau}\sum_{\mu}\mu p_{\mu} + \frac{\beta}{1-\beta}(\bar{\tau}_c + (N-1)\bar{\tau}). \qquad (3.9.5)$$

It is necessary to emphasize that $\bar{t}$ is calculated under the condition that the target is in one of N examined cells and that there are no false lock-ons during the time of searching. If the a priori distribution has a finite width, then fulfillment of the first condition is ensured by examination of all cells, for which $p_{\mu} \neq 0$. Ohterwise $\bar{t}$ is a conditional mathematical expectation, and p_{μ} should be considered as conditional distribution. Selection of the number of examined cells in this case is carried out, proceeding from the permissible probability of target miss not in one of the inspected cells. This probability can be selected a few times less than the permissible probability of nonfulfillment of the tactical problem (for example, not striking the target).

The fulfillment of the second condition is ensured by assignment of a rather low probability of at least one false operation during the time of the search. In systems with automatic tracking, in which false lock-on leads only to losses in time, this condition is replaced by the requirement of a quite prolonged accompaniment of the locked-on target.

On the order of examination of cells in formula (3.9.5), depends only one component

$$\bar{\tau}\sum_{\mu=1}^{N}\mu p_{\mu}, \qquad (3.9.6)$$

It is not difficult to be convinced that this sum has a minimum magnitude, if examination of cells is produced in the order of decrease of a priori probability. For that, it is sufficient to transpose two cells and to consider the change of $\bar{t}$ caused by this transposition

$$\overline{\Delta t} = \bar{\tau}(p_k - p_l)(l - k). \qquad (3.9.7)$$

From (3.9.7) it is obvious that if the cells are arranged in order of decrease $p_{\mu}(p_k > p_l$ at $l > k)$, then any transposition leads to an increase of average time of searching.

In practice, the optimum order of examination in most cases is not used. The exception is, probably, only the case of spiral searching on angles. Usually, the search is made by starting from the edge of the a priori interval, at the time, when with the greatest probability, the target is in the center of the interval. For an estimate of the increase of time of search connected with this, let us consider the particular case of normal a priori distribution. This case is encountered very frequently, since the errors of target designation usually are determined by a whole series of statistically independent factors and the distributive law describing them is near to normal. In considering the variance of error to be great as compared with the size of a unit cell and replacing, in connection with this, the sum by integrals, for a relative increase of average time of search as compared with the optimum case, we obtain at $\beta \ll 1$

$$\gamma = \frac{\Delta \bar{t}}{\bar{t}} = \sqrt{\frac{\pi}{2}}\,\frac{\Phi^{-1}\left(1 - \frac{p_0}{2}\right)}{1 - \exp\left\{-\frac{1}{2}\left[\Phi^{-1}\left(1 - \frac{p_0}{2}\right)\right]^2\right\}} - 1,$$

where $\Phi^{-1}(x)$ is the function, inverse to the integral of probability, and

p_0 is the probability of miss of target in the interval of search.

The character of the dependence $\gamma(p_0)$ is clear from Table 3.1. As can be seen from the table, abandoning the optimum order of examination, can, at small β, lead to multiple losses in time more, the less p_0.

Table 3.1.

p_0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
γ	2.8	3.3	4.1	4.9	5.4	6.25	6,6	7,2	7,7	8,1

Cyclical search with constant number of elements in a cycle is, obviously, not always the best. Actually, in nonuniform a priori distribution and probability of passage different than zero, cases can be shown, when after examining a certain number of cells, the presence of a target in which is very probable, it is better to repeat this examination, than to cross to examination of cells, in which a target with great probability is absent. Thus, we arrive at cyclical searching with a variable number of cells in a cycle. We find the average time of investigation for that case, assuming that every subsequent cycle contains all cells of the preceding cycle.

Let the target be in the μth cell, which is examined, starting from the kth cycle. Then, analogously (3.9.3), we obtain

$$t_l(\mu) = (\tau_c' - \tau) + \mu\tau + (l-k)(\tau_c - \tau) + \tau\sum_{\nu=1}^{l}(l-\nu+1)n_\nu \qquad (3.9.8)$$

where n_ν is the number of cells, examined for the first time in the νth cycle.

In the future by n_ν, will be designated not only the number, but also the entire totality of cells, for the first time examined in the νth cycle. The rule, in accordance with which every cell refers to any totality n_ν, we shall call the distribution of cells between cycles. By averaging (3.9.8) by <u>l</u>, we have

$$\bar{t}(\mu) = (1-\beta)\sum_{l=k}^{\infty}\beta^{l-k}\overline{t_l(\mu)} = (\bar{\tau}_c' - \bar{\tau}) +$$

$$+ \frac{\beta}{1-\beta}(\bar{\tau}_c - \bar{\tau}) + \mu\bar{\tau} + \bar{\tau}(1-\beta)\sum_{l=0}^{\infty}\beta^{l}\sum_{\nu=1}^{l+k}(l+k+1-\nu)n_\nu \qquad (3.9.9)$$

In (3.9.9) k depends on μ. If the μth cell for the first time is examined in the vth cycle $(\mu \in n_\nu)$, then k = v - 1. As a result of averaging by μ, after some conversions, we obtain

$$\bar{t} = (\bar{\tau}_c' - \bar{\tau}) + \frac{\beta}{1-\beta}(\bar{\tau}_c - \bar{\tau}) + \bar{\tau}\sum_{\mu}\mu p_\mu + \\ + \frac{\bar{\tau}}{1-\beta}\sum_{\nu=1}^{\infty} n_\nu \Big\{ \sum_{k=1}^{\nu} p(n_k)\beta^{\nu+1-k} + \\ + \sum_{\nu+1}^{\infty} p(n_k)[(1-\beta)(k-\nu-1)+1]\Big\}. \qquad (3.9.10)$$

where p(n$_k$) is the probability that $\mu \in n_k$.

By minimizing $\bar{t}$, it is possible (in principle, at least) to find the optimum distribution of cells by cycles and optimum order of examination of cells. By comparing (3.95) and (3.9.10), it is not difficult to see that the optimum order of examination is identical for these two cases: the cells should be examined in order of decrease of a priori probability.

Selection of distribution of cells by cycles is significantly more complicated problem, for the solution of which it is useful to introduce into consideration an increase of average time of searching, occurring upon transfer of the th cell from the vth cycle to the (v + 1) th:

$$\frac{\Delta\bar{t}}{\bar{\tau}} = p_\mu\Big(N_\nu + \sum_{\nu+1}^{\infty} n_l\beta^{l-\nu-1} + \beta\Big) - \sum_{1}^{\nu} p(n_k)\beta^{\nu+1-k} - 1 + p(N_\nu). \qquad (3.9.11)$$

where N_ν is the number of cells, examined in the vth cycle $(N_\nu = \sum_{1}^{\nu} n_l)$.

If $\Delta\bar{t} < 0$ at given p_μ, then the transfer of the μth cell to the $(\nu+1)$th cycle leads to a decrease of average time. Proceeding from any distribution of cells by cycles and producing all possible transfers, not increasing the average time of searching, it is possible to arrive at an optimum distribution. The practical realization of this procedure is connected with large calculating difficulties. Therefore, we shall be limited only to the consideration of several very particular cases.

We shall start from the case of finite width of a priori distribution, when $p_\mu = 0$ at $\mu > N$. Let all N cells be distributed between two cycles. Then equality (3.9.11) has the form

$$\sum_{2}^{M} p_\mu [N - (n_1 + 1)(1 - \beta)] \lessgtr 1 + p(n_1)(1 - \beta),$$

from which it follows that to the second cycle it is expedient to refer all those cells, for which

$$p_\mu < \frac{1 - p(n_1)(1 - \beta)}{N - (n_1 + 1)(1 - \beta)}. \tag{3.9.12}$$

From (3.9.12) it follows that if at all μ

$$p_\mu > \frac{\beta}{N - (1 - \beta)} \approx \frac{\beta}{N},$$

then it is better to include all cells already in the first cycle. If the inverse inequality is carried out, then (3.9.12) determines the boundary of the first cycle.

Analogously, if the number of cycles is given and is equal to $\nu + 1$, then the boundary of the νth cycle is determined by the relationship

$$p_\mu < \frac{1 - \sum_{1}^{\nu} p(n_i)(1 - \beta)^{\nu+1}}{N - (n_{\nu+1} + 1)(1 - \beta)}. \tag{3.9.13}$$

If β is small, then, being limited to members of the first order by β and considering $n_{\nu+1} \gg 1$, we can rewrite (3.9.13) in the form

$$p_\mu < \frac{1 - p(N_\nu) + [\beta N_\nu - p(N_\nu, \beta)]}{N\beta + N_\nu(1 - \beta)}. \tag{3.9.14}$$

Relationships (3.9.13), (3.9.14) at $\nu = 1, 2, \ldots$ determine some subdivision of the considered N cells into cycles. This distribution, which is not fully optimum, allows, nevertheless, to receive a very significant gain as compared with the case, when all cells are examined in the first cycle.

As an example, let us consider the case of an exponential a priori distribution, considering for simplicity this distribution to be continuous. Here,

from (3.9.14), we obtain

$$\frac{\beta \ln \frac{1}{p_0}}{1-\beta} + x_\nu \leqslant 1 + \frac{\beta}{1-\beta} e^{x_\nu - x_{\nu-1}}, \qquad (3.9.15)$$

where p_0 is the probability of miss of the target in the examined interval;

$$x = \frac{\mu}{\bar{\mu}},$$

$\bar{\mu}$ is the width of distribution.

Let us assume $p_0 = 10^{-4}, \beta = 10^{-1}$. Here, the boundaries of cycles are determined by the following values of x: $x_1 = 3.45$, $x_2 = 7.65$, $x_3 = 9.2$. The third and subsequent cycles include all examined cells. Corresponding to this case, the average time of searching, calculated by the formula (3.9.10) at $\tau'_c = \tau_c = \tau$, is approximately 30% less than in constant cycles. This gain in time is increased with the increase of probability of pass of the target. Thus, at $\beta = 0.5$ for the considered example, ($x_1 = 2.45$, $x_2 = 5.12$, $x_3 = 7.96$, $x_4 = 9.2$) the gain constitutes 80%.

In the considered example attention is turned to the fact that expansion of cycles occurs evenly ($x_i = ix$). If one were to assume such uniformity from the very beginning, then for exponential a priori distribution it is possible to calculate average time of investigation at arbitrary x, not limiting the total number of examined cells. By making the necessary calculations, from (3.9.10), we obtain

$$\bar{t} = (\tau'_c - \tau) + \frac{\beta}{1-\beta}(\tau_c - \tau) + \mu\tau \times$$
$$\times \left[1 + x\,\frac{[illegible]}{(1-\beta)^2 (1 - e^{-x})^2}\right] \qquad (3.9.16)$$

In Fig. 5.1 is constructed the dependence of $\bar{t}$ on x for various β and at [illegible]. Curves $\bar{t}$ (x) have a minimum, shifting in the direction of large x with the decrease of β. At $\beta \ll 1$, the position and magnitude of the minimum are determined by approximate relationships

$$x \approx \ln\frac{1}{\beta}, \quad \bar{t}_{\min} \approx \mu\tau\left(1 + 2\beta \ln\frac{1}{\beta}\right).$$

The following step on the way of optimization of searching should consist in the rejection of cyclic recurrence. The best order examination will be, obviously, that, with which the solution on which of the cells to inspect is taken after examination of each cell on the basis of information, known a priori and received in preceding inspections. All this information is contained in an a posteriori probability distribution of the presence of a target by the cells. This distribution should be used with the acceptance of the solution as a priori. If in one of cells there occured exceeding of the threshold, then the solution is taken on the presence of a target in this cell and the remaining cells are not examined. Here, the a posteriori probability is equal to a unit in the cell, where the threshold is increased, and to zero in all remaining cells.

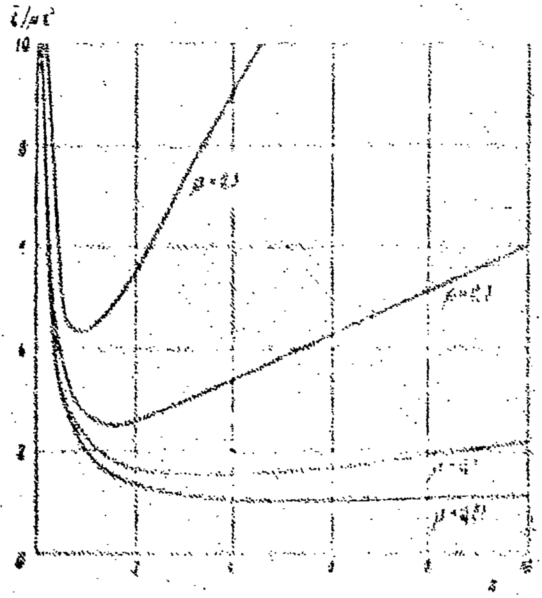

Fig. 3.1. Dependence of average time of searching on the distribution of cells by cycles.

We shall be interested in a posteriori distribution in that case, when not in one of examined cells is a target detected. By the known formula for conditional probability for that distribution we obtain

$$[illegible] \qquad (2.9.17)$$

where P_m is the a priori probability of the presence of a target in the [illegible]th cell;

l_μ is the number of preceding examinations of the μth cell.

The numerator (3.9.17) is an unconditional probability that there is a target in the μth cell and it was missed in all preceding examinations, and the denominator - the probability of the condition under which $p(\mu/l_1, \ldots, l_\mu, \ldots)$, is calculated, i.e., the probability of the absence of exceedings of threshold in all preceding examinations.

It is almost evident that at identical (on the average) delay times in cells, it is best of all to examine them in the order of decrease of a posteriori probability (3.9.17). It is not difficult to show that at any undiminishing function of losses $g(t)$ of such order of examination ensures a minimum of average risk. In order to be convinced of this, we shall calculate the average risk under the condition that the preceding examination conducted for the duration of a certain time did not give results, and subsequent examination starts with a certain μth cell.

In accordance with (3.9.2) and with the assumption on the equality of rights of cells

$$\begin{aligned} R = {} & p(\mu/l_1 \ldots)(1-\beta)\overline{g(t_0+\tau_c')} + \\ & + [1 - p(\mu/l_1 \ldots)](1-\beta)\overline{g(t_0+T+\tau_c')}, \end{aligned} \qquad (3.9.18)$$

where t_0 is the time, expended in the preceding examination;

T is the increase of time of searching, obtained with undetection of the target in the first of the examined cells (i.e., in the μth cell);

τ_c' is the delay time in the cell, where the target is detected.

It is obvious that

$$g(t_0+T+\tau_c') \geq g(t_0+\tau_c'),$$

if $g(t)$ — is an undiminishing function, and consequently, the risk is minimum, when $p(\mu/l_1 \ldots)$ has the highest of the possible values.

In accordance with the received results, optimum searching should be done

in the following manner. We shall arrange the cells in order of decrease of a priori probability p_μ. Examination should start from the first cell and be conducted in order of growth of numbers as long as the inequality is executed

$$p(\mu/1,\overbrace{1,\ldots,1}^{\mu-1},0,\ldots)\geqslant p(1/1,\overbrace{1,\ldots1}^{\mu-1},0,\ldots),$$

i.e., as one may see from (3.9.17), while

$$p_\mu\geqslant p_1\beta. \tag{3.9.19}$$

After in (3.9.19) the inverse inequality starts to be executed (starting from $\mu=\mu_1$), the cells with numbers $\mu\geqslant\mu_1$ are examined alternately with those already inspected, while

$$p_\mu\geqslant p_1\beta^2,$$

after which, the first cells should be examined for the third time alternately with the cells, inspected in the first and second time, etc.

In order to have the possibility to compare such an order of examination with those considered earlier, we shall calculate the average time of search. For that, we shall divide axis μ into intervals, determined by inequalities

$$\beta^l\geqslant\frac{p_\mu}{p_1}>\beta^{l+1}\qquad(l=0;\ 1;\ldots). \tag{3.9.20}$$

The number of cells in each interval will be designated by n_0, n_1,... Each cell $\mu\in n_k$ will be placed in conformity with the totality of cells in other intervals, for which a priori probabilities least of all exceed $p_\mu\beta^{k-s}$ $(s=0,1,\ldots)$. The numbers of these cells, counted off from the beginning of the corresponding intervals, will be designated by μ_0, μ_1... Before we examine for the first time the μth cell, referring to the kth interval, interval n_0 will be examined k times, k - 1 times interval n_1, etc. Furthermore, we shall examine for the (k + 1) th time μ_0 cells of zero interval, for the kth time μ_1 cells of the first interval and so forth to the kth interval. If the target is missed ν times, then the number of examinations of all intervals to the kth is increased

by ν and, furthermore, cells are added, which refer to $n_{k+1}, n_{k+2}, \ldots n_{k+\nu}$. In accordance with what has been said, $\overline{t(\mu)}$ maybe written in the form

$$\overline{t(\mu)} = (\bar{\tau}_c' - \bar{\tau}_c) + \frac{\beta}{1-\beta}(\bar{\tau}_c - \bar{\tau}) + \bar{\tau}\sum_{l=0}^{k}\mu_l + \bar{\tau}\sum_{l=1}^{\infty}\mu_{k+l}\beta^{l}(1-\beta) + (1-\beta)\bar{\tau}\sum_{\nu=0}^{\infty}\beta^{\nu}\sum_{l=0}^{k+l}(k+\nu-l)n_l. \tag{3.9.21}$$

The two first and the last components of the received expression coincide with the corresponding components in formula (3.9.9) for average time of search with expanded cycles. Such a coincidence could have been expected, since the difference between these two methods of searching consists only in the fact that in the case of expanded cycles in the next examination, during which the target is detected, μ - 1 cells are examined, and in the optimum order of examination μ_l cells are inspected of each interval from those already examined by then, i.e., from the first to the (1 + ν)th.

By averaging (3.9.16) by μ and converting the last component, just as in the derivation (3.8.17), we obtain

$$\bar{t} = (\bar{\tau}_c' - \bar{\tau}) + \frac{1}{1-\beta}(\bar{\tau}_c - \bar{\tau}) + \bar{\tau}\sum_{\mu}p_{\mu}\Big[\sum_{l=0}^{k(\mu)}\mu_l + (1-\beta)\sum_{l=k(\mu)+1}^{\infty}\mu_l\beta^{l-k(\mu)}\Big] + \frac{\bar{\tau}}{1-\beta}\sum_{\nu=1}^{\infty}n_{\nu}\Big\{\sum_{1}^{\nu}p(n_k)\beta^{\nu+1-k} + \sum_{\nu+1}^{\infty}p(n_k)[(1-\beta)(k-\nu-1)+1]\Big\}. \tag{3.9.22}$$

In particular, for exponential a priori distribution

$$\bar{t} = (\bar{\tau}_c' - \bar{\tau}) + \frac{\beta}{1-\beta}(\bar{\tau}_c - \bar{\tau}) + \bar{\mu}\bar{\tau}\,\frac{1+\beta\ln\frac{1}{\beta} - 2\beta^2 + \beta^2\ln\frac{1}{\beta} + \beta^3}{(1-\beta)^2}. \tag{3.9.23}$$

At $\beta \ll 1$, the factor at $\bar{\mu}\bar{\tau}$ in (3.9.23) is approximately equal to

$$1 + \beta\ln\frac{1}{\beta}.$$

and $\bar{t}$ is very close to the average time of search in the case of expanded cycles at optimum selection of increase of $x = \frac{n}{\mu}$.

The noted proximity of average times of search in expanded cycles of optimum duration and with optimum order of examination shows that the method of expanded cycles can, without significant losses in time, be used instead of the optimum method, the realization of which is connected with large technical difficulties.

This result can be used also in the selection of duration of expanded cycles. Considering the smallness of the difference of average times for the two shown methods, it is possible to expect that subdivision into cycles in accordance with (3.9.20) is close to optimum.

Above were considered various cases of selection of order of examination of cells, ensuring minimum of average time of search. In some practical problems, it is more justified to find an optimum method of target search, proceeding from the requirement of maximum probability of detection (minimum probability of miss) of target in a certain fixed time. To solve this problem is possible only for the case of fixed time of delay in a cell. The received results can be, with some effort, used in random delay, if the total time, expended for detection, is great as compared with the average delay time in the cell.

Let T be the time expended in the search, and τ be the delay time in the cell. If, during time T, a certain μth cell is examined l_μ times and the signals from the target in various examinations are independent, then for the probability of detection of target it is possible to write

$$D = \sum_{\mu} p_{\mu} (1 - \beta^{l_{\mu}}) = 1 - \sum_{\mu} p_{\mu} \beta^{l_{\mu}}. \qquad (3.9.24)$$

It is necessary to select l_μ in such a manner that D is maximum and the condition is fulfilled

$$\tau \sum_{\mu} l_{\mu} = T. \qquad (3.9.25)$$

We shall assume that the method of search, which is, for the considered case,

optimum, is characterized by the numbers $l_1, l_2, \ldots,$ satisfying condition (3.9.25). We shall replace in (3.9.24) $l_k = 0$ by $l_k' = l_k + 1$, and $l_i \geqslant 1$ by $l_i' = l_i - 1$, preserving the remaining l_j constant. With such replacement, condition (3.9.25) remains, obviously, in force, and probability D changes to the magnitude

$$\Delta D = (1 - \beta)(p_k \beta^{l_k} - p_i \beta^{l_i}). \qquad (3.9.26)$$

Inasmuch as numbers $l_1, l_2, \ldots,$ correspond to optimum search, ΔD should be negative at any i and k, only if $l_i \geqslant 1$. It follows from this that for all i, for which $l_i > 0$, the product of $p_i \beta^{l_i}$ should be approximately constant, i.e.,

$$l_i \approx \ln \frac{C}{p_i} \Big/ \ln \beta. \qquad (3.9.27)$$

The magnitude of constant C is determined by condition (3.9.25). An approximate formula (3.9.27) is carried out more accurately, the larger the $\frac{T}{\tau}$. At $\frac{T}{\tau} \gg 1$, it is possible to consider approximately $l_i = 0$ at $p_i \leqslant C$. With these assumptions, the calculation of optimum distribution of time T between cells does not present much effort. With small $\frac{T}{\tau}$, the calculation is brought to an end by only numerical methods.

As example, let us consider normal a priori distribution p_μ with dispersion σ^2, equal to the variance of error of target designation, divided by the width of the unit cell.

Considering $\sigma \gg 1$ and $\frac{T}{\tau} \gg 1$, we shall replace all sums by integrals, and μ_0, starting from which $l_\mu = 0$, we shall determine from the condition $p_\mu = C$. Here,

$$\mu_0 = \sqrt{2\sigma^2 \ln \frac{1}{\sqrt{2\pi}\sigma C}}$$

and in accordance with (3.9.27)

$$l_\mu \approx \frac{1}{2 \ln \frac{1}{\beta}} \left(\frac{\mu_0^2}{\sigma^2} - \frac{\mu^2}{\sigma^2} \right).$$

Substituting this expression in (3.9.25), we obtain

$$\mu_0 \approx \sigma \sqrt[3]{\frac{3T}{2\tau\sigma} \ln \frac{1}{\beta}}$$

and

$$l_\mu \approx \frac{1}{2 \ln \frac{1}{\beta}} \left[\left(\frac{3}{2} \frac{T}{\tau\sigma} \ln \frac{1}{\beta} \right)^{2/3} - \frac{\mu^2}{\sigma^2} \right]. \qquad (3.9.28)$$

Using formula (3.9.28), it is simple for specific β, T, τ and σ to find the optimum distribution of time T between cells, ensuring maximum probability of detection of target during that time. In distinction from the case of minimization of average time of search, this optimum distribution does not determine the order of examination of cells. It can be realized, for example, with the help of cyclical search with expanded cycles. Here, the limits of search and the distributive law of cells by cycles should be determined by formula (3.9.28).

The considered problems, undoubtedly, do not by far exhaust all problems, connected with optimization of search and scanning. In particular, the problems of search were hardly considered in a number of targets more than one. The very interesting problem optimization of scanning, based on the requirement of most rapid detection of randomly appearing targets was not touched upon.

In connection with the development of methods of processing radar information, obtained during scanning, with the help of electronic computers there appeared one more very important and interesting problem, formulated in sufficiently general form in [52]. This problem reduces to optimization of the method of filling a finite number of machine memory tracks with data on the trajectory of the assumed targets. An optimum method should ensure the fastest detection of a true target after its appearance. Unfortunately, any results in this area, as far as it is known by the authors, are presently lacking.

3.10. Conclusion

The conducted consideration of methods of acceptance of solutions on the

presence and position of a target, optimum from various points of views during detection showed that under very general conditions these methods reduce to the comparison, with the threshold and among themselves, of the values of the relation of verisimilitude for all possible a priori values of parameters of the detected target. Here, the system of detection consists of a certain number of channels, each of which is intended for processing a signal from the target with the fully determined values of parameters. This circumstance considerably facilitates analysis and synthesis of systems of detection and will allow us, in Chapters 4 and 5, to consider, mainly, the case of detection of a target with known parameters, applying the received results to multichannel systems. An estimate of parameters of the detected target, as showed the consideration, should be made on the basis of comparison of the values of a posteriori probability of the presence of a target in various points of the a priori interval or region. There is a basis to consider that when the permissible error of measurement of parameter coincides with or exceeds the interval of solution by this parameter, then such an operation is equivalent to the selection an estimate of that value of λ, with which $p_0(\lambda)\Lambda(y,\lambda)$ exceeds the threshold. A comparison of these two methods for particular cases will be conducted in subsequent chapters.

In the future we shall be limited to the case of fixed time of observation at each stage of detection, not concerning systems, using the procedure of sequential analysis. As already was noted in Section 3.5, a sufficiently complete investigation of these systems, which would consider the actual properties of fluctuations of a reflected signal and would include the case of detection of a target with unknown coordinates, cannot be made with the existing methods.

In sections 3.8 and 3.9 were formulated certain problems, concerning optimization of space spanning and target search. The solution of these problems was found only at that stage, at which it turns out to be necessary to use the specific dependence of probability of miss from time of observation and

probability of false alarm. After these dependence are found in subsequent chapters, we will continue the solution of the indicated problems for those particular cases, for which this can be done without the use of laborious numerical methods. Some results of Sections 3.8 and 3.9 present independent interest. Here, are included, basically, the results on optimization of the order of examination of cells during search.

It turned out that in the absence of any limiting assumptions, the examination of cells in the order of decrease of a posteriori probability of the presence of a target is optimum. If the cells are located in order of decrease of a priori probability of the presence of a target, then the optimum order of examination is close in character and in effectiveness to cyclical examination with optimum distribution of cells between cycles. This method of examination, apparently, in the majority of systems is easier to carry out technically.

CHAPTER 4

DETECTION OF A COHERENT SIGNAL

4.1. Introductory Remarks

In this and subsequent chapters, the general results of chapter 3 will be used for the synthesis of optimum systems of detection of coherent and incoherent signals on the background of noises and certain forms of interferences. Along with synthesis, we will conduct analysis of existing systems of detection, and also quasioptimum systems, obtained from optimum ones by means of their simplification. The utilized division into coherent and incoherent signals, as already was noted in Chapter 1, is connected exclusively with considerations of convenience of solution of problems of analysis and synthesis. In accordance with this division, the incoherent signals, considered in the following chapter, refer to signals with periodic modulation, for which the initial phases of high-frequency oscillations in various periods change independently and randomly, being subordinated to the uniform distributive law of probability in interval $(0,2\pi)$. Signals, for which such uncontrollable jumps of phase are absent, we shall call coherent.

In accordance with the results of Chapter 1, a signal, reflected from a target, will be, in an overwhelming majority of cases, considered as a normal random process.

Synthesis of optimum, and also analysis of a broad class of nonoptimum systems of detection with a normal received signal y(t), i.e., with normal distribution for the signal from the target and for interference, can be conducted on the basis of some general relationships, determining the character of optimum operations and the form of the characteristics of detection. These relationships were, for the first time, received in [53]. In Section 4.2, their derivation will be conducted by a somewhat simpler method (in the part, concerning the obtainment of characteristics of detection). In subsequent paragraphs, the general relationships will be used for investigation of questions of detection of a radar signal on the background of noises and interferences, subordinated by normal distributive law (passive and wide-band noise interferences, which is normalized upon passage through input filters of the receiver).

Examination of these questions is expedient to start from the most frequently encountered case of detection on the background of noises. Investigation of methods of maximum increase of reliability of detection of weak signals on the background of internal noises of the receiver and external thermal noises has a direct relation to the problem of increase of range of radar stations and presents great practical interest. This problem is devoted to a large number of published works, from which, however, only a comparatively small part is carried out with respect to fluctuations of the reflected signal [7, 9, 19, 40]. Calculation of fluctuations in these works was reduced to consideration of extreme cases: very slow (a pack of harmoniously fluctuating signals) and very fast (a pack independently fluctuating signals) fluctuations. In this work, the question of the influence of fluctuations on reliability of detection is considered more specifically.

Wide-band noise interference, by the character of its interfering influence, in a significant degree is equivalent to internal noise. The peculiarities of the influence of interferences are connected basically with the fact that

its spectral density is unknown and that the interference is modulated by intensity during scanning. The influence of these peculiarities, and also the possibilities of protection from interferences, will be considered in Section 4.12. In the same place will be considered the question of the influence on systems of detection of a coherent signal of pulse chaotic interference.

Considerable attention in this chapter is given to the important problem of noise-resistance of systems of detection in reference to passive interferences. Synthesis is conducted for optimum systems of detection of a signal on the background of noises and passive interferences for an arbitrary law of modulation, and also synthesis of optimum intra- and interperiod processing for the case of separate processing of periods of the signal. On the basis of results of synthesis, is conducted a comparison of various laws of modulation from the point of view of ensured noise-resistance and the criteria are worked out for the selection of the law of modulation. Optimum methods of signal processing are compared with the method of period-by-period compensation, applied for suppression of interfering reflections from motionless objects in pulse radar stations [1, 59, 65, 66].

Questions of synthesis of optimum systems of detection in the presence of passive interferences were considered earlier in [9, 54--58, 64], the results of which are partially used here.

4.2. Relation of Verisimilitude and Its Statistical Characteristics for Gauss Signal and Interference

As already was noted in Chapter 1, the received signal, which is the sum of the coherent signal, reflected from the target, and interferences, for a very broad class of cases can be considered as a normal random process and be characterized by the functional of probability density

$$F[y(t)] = K_0 \exp\left\{-\frac{1}{2}\int_0^T\int_0^T W(t_1, t_2)\, y(t_1)\, y(t_2)\, dt_1\, dt_2\right\}. \qquad (4.2.1)$$

where K_0 is a coefficient not depending on realization of the received signal $y(t)$;

$W(t_1, t_2)$ is the function which is a continuous matrix analog, inverse to the correlation matrix.

Function $W(t_1, t_2)$ is connected with the correlation function by equation (1.4.2).

The relation of verisimilitude is expressed by the formula

$$\Lambda(y) = \frac{K_{0\text{с}}}{K_{0\text{н}}} \exp\left\{ \frac{1}{2} \int_0^T\int_0^T [W_{\text{н}}(t_1, t_2) - W_{\text{сн}}(t_1, t_2)] \times \right.$$
$$\left. \times\, y(t_1)\, y(t_2)\, dt_1\, dt_2 \right\}. \qquad (4.2.2)$$

where $W_{\text{н}}(t_1, t_2)$ is the solution of equation (1.4.2) at $R(t_1, t_2)$, equal to the function of correlation of interference $R_{\text{н}}(t_1, t_2)$;

$W_{\text{сн}}(t_1, t_2)$ is the solution of the same equation at $R(t_1, t_2)$, equal to the function of correlation of mixture of signal and interference

$$R_{\text{сн}}(t_1, t_2) = R_{\text{с}}(t_1, t_2) + R_{\text{н}}(t_1, t_2).$$

Combining the equations for $W_{\text{н}}(t_1, t_2)$ and $W_{\text{сн}}(t_1, t_2)$, it is easy to show that function

$$V(t_1, t_2) = W_{\text{н}}(t_1, t_2) - W_{\text{сн}}(t_1, t_2), \qquad (4.2.3)$$

in (4.2.2), should satisfy equation

$$\int_0^T V(t_1, t)[R_{\text{н}}(t, t_2) + R_{\text{с}}(t, t_2)]\, dt = \int_0^T W_{\text{н}}(t_1, t)\, R_{\text{с}}(t, t_2)\, dt. \qquad (4.2.4)$$

Inasmuch as the coefficient in front of the exponent in (4.2.2) does not depend on the accepted realization, one can, obviously, take the solution on the presence of the target on the basis of comparison with the corresponding threshold of the functional

$$L[y] = \frac{1}{2} \int_0^T\int_0^T V(t_1, t_2)\, y(t_1)\, y(t_2)\, dt_1\, dt_2. \qquad (4.2.5)$$

In order to explain the character of operations on the received signal, leading to the formation of L[y], it is necessary to find, in evident view, function V (t_1, t_2) from equations (1.4.2) and (4.2.4). The solution of these equations reduces, thus, to the problem of synthesis of an optimum system of detection.

Some conclusions relative to the possible methods of realization of these operations can be made without solving the indicated equations. From the symmetry of the function of correlation and equations for V (t_1, t_2) it follows that this function also is symmetric, i.e.,

$$V(t_1, t_2) = V(t_2, t_1).$$

By dividing the integral by t_2 in (4.2.5) into two (from 0 to t_1 and t_1 to T) and using the indicated property of V (t_1, t_2), it is simple to reduce (4.2.5) to the form

$$L[y] = \int_0^T y(t_1)\,dt_1 \int_0^{t_1} V(t_1, t_2)\, y(t_2)\,dt_2.$$

In this formula, the function V (t_1, t_2) may be considered as the pulse reaction of a linear filter. Thus, the formation of L[y] reduces to the transmission of the received signal through the filter, multiplication of the received signal on the output again by the received signal and integration.

Fig. 4.1. Functional diagram of optimum system of detection: 1) filter with pulse reaction V (t_1, t_2); 2) integrator during time T; 3) relay.

Corresponding to these operations, the block-diagram of the receiving installation is shown in Fig. 4.1. Such an interpretation of optimum operations was proposed in [53].

For obtaining the characteristics of detection, corresponding to the comparison of L[y] with the threshold, it is necessary to know the distributive law of magnitude L in the presence of and absence in y(t) of a signal, reflected from the target. Calculation of this law in clear form requires, undoubtedly, assignment of functions of correlation of interference and useful signal. In a general form we can obtain only an expression for the characteristic function, corresponding to this distributive law. In order to do this, we temporary replace integrals in (4.2.5) by the sums

$$L_n[y]=\frac{1}{2}\sum_{j,k=1}^{n} V(j\Delta t, k\Delta t)\, y(j\Delta t)\, y(k\Delta t)(\Delta t)^2,$$

where $n=\left[\frac{T}{\Delta t}\right]$ — is the integral part of relation $\frac{T}{\Delta t}$.

The joint distribution of magnitudes $y(j\Delta t)$ $(j=1, \dots, n)$ has the form

$$\frac{1}{(2\pi)^{n/2}\sqrt{|R(j\Delta t, k\Delta t)|}}\exp\left\{-\frac{(\Delta t)^2}{2}\sum_{j,k=1}^{n} W(j\Delta t, k\Delta t)\times \right.$$
$$\left. \times y(j\Delta t)\, y(k\Delta t)\right\},$$

where $R(t_1,t_2)$ is the function of correlation of the received signal; $W(t_1,t_2)$ is the corresponding solution of equation (1.4.2).

For the characteristic function of magnitude $L_n[y]$ we have

$$\Psi_n(\eta)=\overline{e^{i\eta L[y]}}=\frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{|R(j\Delta t, k\Delta t)|}}\times$$
$$\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\sum_{j,k=1}^{n}[W(j\Delta t, k\Delta t)-i\eta V(j\Delta t, k\Delta t)]\times\right.$$
$$\left.\times(\Delta t)^2 y(j\Delta t)\, y(k\Delta t)\right\}dy(\Delta t)\dots dy(n\Delta t)=$$
$$=\left\{|R(j\Delta t, k\Delta t)|\,|W(j\Delta t, k\Delta t)(\Delta t)^2-i\eta V(j\Delta t, k\Delta t)(\Delta t)^2|\right\}^{-\frac{1}{2}}=$$
$$=\left\{\left|\delta_{jk}-i\eta(\Delta t)^2\sum_{l=1}^{n} R(j\Delta t, l\Delta t)\, V(l\Delta t, k\Delta t)\right|\right\}^{-\frac{1}{2}}.$$

For calculation of this determinant, it is possible to use the method,

described in Section 1.4. As a result, we obtain

$$\Psi(\eta)=\lim_{n\to\infty}\Psi_n(\eta)=\exp\left\{-\frac{1}{2}\int_0^T dt\int_0^{i\eta} G(t, t; \lambda)\,d\lambda\right\}, \qquad (4.2.6)$$

where function $G(t_1, t_2; \lambda)$ is determined from the equation

$$G(t_1, t_2; \lambda)-\lambda\int_0^T G(t_1, t; \lambda)V_1(t, t_2)\,dt=V_1(t_1, t_2), \qquad (4.2.7)$$

and

$$V_1(t_1, t_2)=\int_0^T V(t_1, t)R(t, t_2)\,dt. \qquad (4.2.8)$$

One should note that in the derivation of the formula for the characteristic function we did not assume that function $V(t_1, t_2)$ is determined by equation (4.2.4) and corresponds to optimum processing. In connection with this, this formula can be also used for calculation of characteristics of detection, corresponding to nonoptimum processing, if its result can be represented in the form of (4.2.5), with arbitrary function $V(t_1, t_2)$.

Frequently, the result of processing the signal can be presented in quadratic form from complex functions $f(t)$

$$L(f)=\int_0^T\int_0^T v(t_1, t_2)f(t_1)f^*(t_2)\,dt_1dt_2, \qquad (4.2.5')$$

where the real and imaginary part of $f(t)$ are distributed by normal law with zero mean values, and

$$v(t_2, t_1)=v^*(t_1, t_2).$$

The derivation of the characteristic function of magnitude $L(f)$ is fully analogous to that just now given. Joint distribution for real and imaginary parts of magnitudes $f(j\Delta t)$ can be written in the form

$$\frac{1}{\pi^n\,|r(j\Delta t, k\Delta t)|}\exp\left\{-\sum_{j,k} r^{-1}(j\Delta t, k\Delta t)f(j\Delta t)f^*(k\Delta t)\right\}.$$

where $r(t_1, t_2)=\overline{f(t_1)f^*(t_2)}$, and $\|r^{-1}(j\Delta t, k\Delta t)\|$ — is the inverse $\|r(j\Delta t, k\Delta t)\|$ of matrix.

Replacing, in the expression for L(f), the integrals by sums, calculating the characteristic function by direct integration and changing to the limit at $\Delta t \to 0$, analogous to (4.2.6) we obtain

$$\Psi(\eta) = \exp\left\{\int_0^T dt \int_0^{i\eta} G_1(t, t; \lambda)\, d\lambda\right\}, \tag{4.2.6'}$$

where $G_1(t_1, t_2; \lambda)$ is determined from the equation

$$G_1(t_1, t_2; \lambda) - \lambda \int_0^T G_1(t_1, t; \lambda)\, v_1(t, t_2)\, dt = v_1(t_1, t_2), \tag{4.2.7'}$$

and

$$v_1(t_1, t_2) = \int_0^T v(t_1, t)\, r(t, t_2)\, dt. \tag{4.2.8'}$$

The received relationships can be easily generalized in the case, when the solution on the presence of a target is taken on the basis of several signals $y_1(t)$, $y_2(t), \ldots, y_m(t)$, the joint distributive law of which is Gauss, and the mean value is equal to zero. This case is encountered, for example, in multi-frequency operation. Here,

$$L_0(y) = \frac{1}{2} \sum_{j,k=1}^{m} \int_0^T \int_0^T V_{jk}(t_1, t_2)\, y_j(t_1)\, y_k(t_2)\, dt_1\, dt_2, \tag{4.2.9}$$

where function $V_{jk}(t_1, t_2)$ is determined by equations

$$\begin{aligned} \sum_{l=1}^{m} \int_0^T V_{jl}(t_1, t)\left[R_{clk}(t, t_2) + R_{nlk}(t, t_2)\right] dt = \\ = \sum_{l=1}^{m} \int_0^T W_{njl}(t_1, t)\, R_{clk}(t, t_2)\, dt, \end{aligned} \tag{4.2.10}$$

$$\sum_{l=1}^{m} \int_0^T W_{njl}(t_1, t)\, R_{nlk}(t, t_2)\, dt = \delta_{jk}\,\delta(t_1 - t_2). \tag{4.2.11}$$

The characteristic function in a multi-dimensional case is determined by these relationships:

$$\Psi_0(\eta) = \exp\left\{\frac{1}{2} \sum_{j=1}^{m} \int_0^{i\eta} d\lambda \int_0^T G_{jj}(t, t; \lambda)\, dt\right\}. \tag{4.2.12}$$

$$G_{jk}(t_1, t_2; \lambda) - \lambda \sum_{l=1}^{m} \int_0^T G_{jl}(t_1, t; \lambda) V'_{lk}(t, t_2)\, dt = V'_{jk}(t_1, t_2), \tag{4.2.13}$$

$$V'_{jk}(t_1, t_2) = \sum_{l=1}^{m} \int_0^T V_{jl}(t_1, t) R_{lk}(t, t_2)\, dt. \tag{4.2.14}$$

Analogously generalized are relationships (4.2.6) and (4.2.8).

Substituting in these formulas the functions of mutual correlation of components of the received signal

$$R_{lk}(t_1, t_2) = \overline{y_l(t_1) y_k(t_2)}$$

in the presence and absence of a target, it is possible, in principle, by the found expressions for characteristic functions, to find the corresponding distributive laws of magnitude L_0 and to obtain an expression for probabilities of miss and false alarm.

The relationships for the multi-dimensional case are considerably simplified, if the received signals $y_1(t), \ldots, y_m(t)$ turn out to be statistically independent (for example, due to large detuning of the utilized frequency channels). Here, the joint distribution of probability of signals is equal to the product of distributions for separate signals. In accordance with this, functional $L_0(y)$ turns out to be equal to the sum of statistically independent functionals for separate signals

$$L_0(y) = \sum_{j=1}^{m} L_j(y), \tag{4.2.15}$$

and the characteristic function of random variable $L_0(y)$ is equal to the product of characteristic functions $\Psi_j(\eta)$.

4.3. Optimum Detection of Signal in Noise

4.3.1. General Relationships, Determining the Form of Optimum Processing

Let us consider the most frequently arising problem of detection of a radar signal on the background of noises with uniform spectral density N_0. The function of the correlation of received signal in this case is determined by equalities (see Section 1.4)

$$R_{\text{п}}(t_1, t_2) = N_0 \delta(t_1 - t_2), \tag{4.3.1}$$

$$R_{\text{сп}}(t_1, t_2) = N_0 \delta(t_1 - t_2) + \\ + P_{\text{c}} \operatorname{Re} u(t_1 - \tau) u^*(t_2 - \tau) \rho(t_1 - t_2) e^{i\omega_1(t_1 - t_2)}, \tag{4.3.2}$$

where $\omega_1 = \omega_0 + \omega_{\text{д}}$ — is the carrier frequency of the received signal.

It is not difficult to be convinced that function $W_{\text{п}}(t_1, t_2)$, corresponding to (4.3.1), has the form

$$W_{\text{п}}(t_1, t_2) = \frac{1}{N_0} \delta(t_1 - t_2). \tag{4.3.3}$$

Usually fluctuation of the signal, the speed of which is characterized by the speed of decrease of function $\rho(t)$, are slow as compared with the law of modulation u(t) and do not distort the law of modulation of the received signal. Fulfillment of this condition is necessary for effective use of modulation during detection and measurement of parameters of a target. In the future, we shall consider fluctuation to be quite slow, assuming, during periodic modulation, $\rho(t)$ to be slightly changing in the period, and in the case of complex-modulated single sendings, slightly changing in the duration of sending. Here, it is natural to expect that optimum processing of a fluctuating signal will include multiplication by the expected signal, which is the main element of optimum processing without fluctuations [2, 8], and find the solution of equation (4.2.4) in the form

$$V(t_1, t_2) = \operatorname{Re} v(t_1, t_2) u(t_1 - \tau) u^*(t_2 - \tau) e^{i\omega_1(t_1 - t_2)}. \tag{4.3.4}$$

considering function $v(t_1, t_2)$ just as slow, as $\rho(t_1, t_2)$. A more formal basis for finding the solution in such form is the circumstance that at $\rho(t) = 1\,(|t| < T)$ the equation (4.2.4) has a degenerate nucleus $R_c\,(t_1, t_2)$ and the solution of the form (4.3.4) with $v(t_1, t_2)$ = const. With a slow change of $\rho(t)$, the nucleus of equation (4.2.4) is quasidegenerate and it is natural to assume that its solution in its form is also similar to degenerate.

Substituting (4.3.2) and (4.3.4) in (4.2.4), we obtain

$$v(t_1, t_2) + \frac{P_c}{2N_0}\int_0^T |u(t-\tau)|^2 v(t_1, t)\,\rho(t-t_2)\,dt = \frac{P_c}{N_0^2}\,\rho(t_1 - t_2). \qquad (4.3.5)$$

We shall now use the assumption made above on the slowness of fluctuations as compared with the law of modulation of the main signal. In the case when the signal is a complex-modulated single sending, this assumption leads to replacement of $\rho(t_1 - t_2)$ in (4.3.5) by one. The solution of equation (4.3.5) in this case is constant (see paragraph 4.3.3). If the law of modulation is periodic or represents a stationary random process, then, by using the slowness of fluctuations, it is possible, under the integral, to average $/u(t)/^2$ in time, whereby in accordance with the utilized normalization (Section 1.2),

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |u(t)|^2\,dt = 1.$$

Then (4.3.5) obtains the following form:

$$v(t_1, t_2) + \frac{P_c}{2N_0}\int_0^T v(t_1, t)\,\rho(t-t_2)\,dt = \frac{P_c}{N_0^2}\,\rho(t_1 - t_2). \qquad (4.3.5')$$

Not specifying $\rho(t)$, the solution of (4.3.5') can be obtained for two extreme cases, when $\rho(t)$ diminishes during time $\tau_{\text{ж}c}$ (time of correlation), significantly less than T, and when $\rho(t)$ practically does not change during time T.

4.3.2. Case of Fast Fluctuations

At $\tau_{\text{ж}c} \ll T$, by disregarding the extreme effects, it is possible to consider the limits of integration in (4.3.5') to infinite and to solve this equation by

Fourier transform. As a result, we obtain

$$v(t_1, t_2)=\frac{1}{\pi N_0}\int_{-\infty}^{\infty}\frac{hS_0(\omega)\,e^{i\omega(t_1-t_2)}}{1+hS_0(\omega)}\,d\omega, \tag{4.3.6}$$

where $h = \frac{P_c}{2N_0\Delta f_c}$;

Δf_c — is the effective width of the spectrum signal fluctuations;

$S_0(\omega)$ — is spectral density of fluctuations, standardized so that its maximum is equal to one.

Substituting (4.3.4), (4.3.6) in (4.2.5) and using the Parseval theorem, we have

$$L(y)=\frac{1}{2\pi N_0}\int_{-\infty}^{\infty}\frac{hS_0(\omega)}{1+hS_0(\omega)}|Y_1(\omega)|^2\,d\omega=\frac{1}{N_0}\int_0^T|Q(t)|^2\,dt, \tag{4.3.7}$$

where $Y_1(\omega)$ is the spectrum of the product of observed realization $y(t)$ by $u(t-\tau)e^{i\omega_0 t}$,

$Q(t)$ is the result of transmission of this product through a filter, the square of the modulus of frequency response of which is equal to

$$|H_0(i\omega)|^2=\frac{hS_0(\omega)}{1+hS_0(\omega)}. \tag{4.3.8}$$

Optimum processing of the received signal, determined by formulas (4.3.7), (4.3.8), consists in multiplication of the received signal by the expected signal $u(t-\tau)e^{i\omega_0 t}$, transmission through a filter with frequency response $H_0(i\omega)$, formation of the square of the modulus of output voltage of the filter and integration during time T.

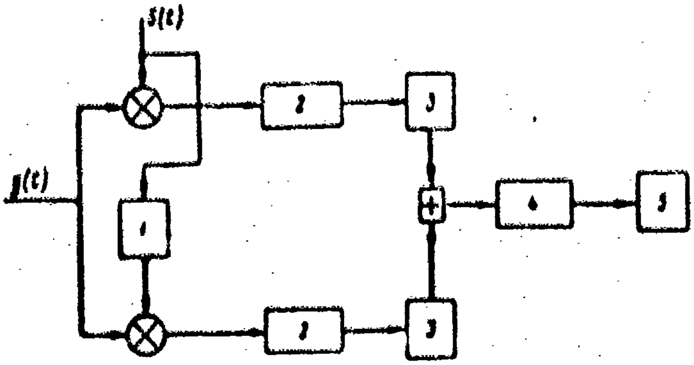

Fig. 4.2. Functional diagram of optimum system of detection for the case of fast fluctuations with multiplication by the reference signal $s(t)=u_0(t-\tau)\cos[\omega_0 t+\psi(t-\tau)]$ and filtration at low frequency: 1) phase inverter at 90°; 2) narrow-band filter; 3) square-law function generator ($u_{output} = a u^2_{input}$); 4) integrator during time T; 5) relay.

The block-diagram, carrying out these operations, is shown in Fig. 4.2. Heterodyning and transmission through the filter are carried out in this diagram in two quadrature channels. The output voltages of the filters are raised in the square and are added, the received sum is integrated by interval of time T and compared then with the threshold of relay, selected in accordance with the given probability of false alarm.

In most cases, it is technically more convenient to carry out filtration at intermediate frequency and to use, for obtaining the square of the modulus of complex oscillation (square of envelope), a squarelaw detector. In connection with this, it is desirable to add optimum operations of such form, which would allow their realization in a similar diagram. We shall displace reference signal $u_u(t-\tau)\cos[\omega_1 t+\psi(t-\tau)]$ in frequency to a magnitude of some intermediate frequency W_{np}, to which the filter is tuned.

The pulse reaction of such a filter will be registered in the form

$$h(t)\cos\omega_{\text{пр}}t,$$

where h(t) is the pulse reaction of the filter $H_0(i\omega)$.

The square of the envelope of voltage on the output of the filter is recorded in the form

$$\begin{aligned}&\Big[\int_0^t y(t_1)u_u(t_1-\tau)\cos[(\omega_1-\omega_{\text{пр}})t_1+\psi(t_1-\tau)]\times\\&\qquad\times h(t-t_1)\cos\omega_{\text{пр}}t_1dt_1\Big]^2+\\&+\Big[\int_0^t y(t_1)u_u(t_1-\tau)\sin[(\omega_1-\omega_{\text{пр}})t_1+\psi(t_1-\tau)]\times\\&\qquad\times h(t-t_1)\sin\omega_{\text{пр}}t_1dt_1\Big]^2.\end{aligned}$$

By replacing the product of cosines and sines by cosines of the sum and difference, it is simple to be convinced that this expression coincides with $Q(t)\ ^2$, if the signal on the input of the mixer y(t) does not contain frequencies, close to $\omega_1-2\omega_{\text{пр}}$, i.e., if the input circuits of the receiver ensure sufficient suppression of image frequency. This condition is usually fulfilled,

so that both considered versions of the optimum diagram can be considered fully equivalent. The block-diagram with filtration by intermediate frequency is shown in Fig. 4.3.

We shall pause briefly on the physical meaning of optimum operations. Multiplication of the received signal by the expected law of modulation of the signal from the target ensures suppression of noise components, not coinciding in form with the expected law of modulation (in any degree orthogonal to this law). Subsequent filtration ensures suppression of noise components, not coinciding with the expected signal in frequency.

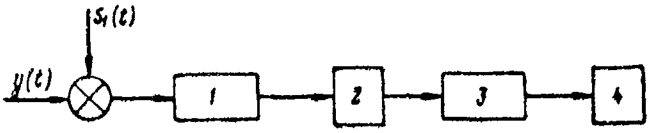

Fig. 4.3. Functional diagram of optimum system of detection for the case of fast fluctuations with filtration on intermediate frequency, $s_1(t) = u_1(t-\tau)\cos[(\omega_1 - \omega_{пр})t + \psi(t-\tau)]$: 1) filter; 2) detector; 3) integrator during Time T; 4) relay.

This filtration can be treated also as coherent accumulation of the signal during time, comparable in order magnitude with time of correlation of fluctuations, i.e., during the time, for the duration of which the coherence is maintained. Incoherent post-detector accumulation is carried out in a large time interval T, in which coherent accumulation is impossible due to fluctuations.

The form of frequency response $H_0(i\omega)$ and transmission band of filter depend on the form of the spectrum fluctuations of the reflected signal and on the relation h of power of the signal to the power of noise in the band $\Delta f_э$. At small h the function $|H_0(i\omega)|^2$ coincides in form with the spectral density of fluctuations, and the filter band with the width of the spectrum of fluctuations. If the spectrum of fluctuations can, with sufficient accuracy, be approximated by a rectangle, then such a coincidence takes place at all h. At other

$S_0(\omega)$, the filter band is expanded with the increase of h, faster, the slower $S_0(\omega)$ drops with the increase of $|\omega|$. Thus, at

$$S_0(\omega)=\frac{1}{1+\left(\frac{\omega}{2\Delta f_c}\right)^2}$$

the effective band of transmission of the filter

$$\Delta f_\Phi=\sqrt{1+h}\,\Delta f_c,$$

at

$$S_0(\omega)=\frac{1}{\left[1+\left(\frac{\omega}{4\Delta f_c}\right)^2\right]^2}$$

the effective transmission band

$$\Delta f_\Phi=\Delta f_c\sqrt{\frac{2(1+h)}{\sqrt{1+h}+1}}\underset{h\to\infty}{\to}\sqrt{2}\,\Delta f_c\sqrt{1+h}.$$

For a Gauss spectrum with the same effective band, the band of transmission on the level of half power (effective band cannot be expressed by h in clear form) is equal to

$$\Delta f_{\Phi 0,5}=\Delta f_c\sqrt{\frac{2}{\pi}\ln(1+h)}.$$

It is necessary to note that expansion of the filter band with the growth of the signal-to-noise ratio h is justified not only purely by power considerations, which are reduced to the tendency to miss a whole section of the spectrum, where spectral density signal fluctuations is greater than spectral density of noise. In expansion of the filter band along with the increase of transmitted power, there occurs an increase of the number of independent values of noise, summarized during subsequent incoherent accumulation. This promotes the decrease of relative magnitude of fluctuations of output voltage. A decrease of time of coherent accumulation of signal and signal-to-noise ratio on the output of the filter limit expansion of the band and ensure the existence of an optimum.

As can be seen from the received results, optimum processing during fast

fluctuation depends on the signal-to-noise ratio. This creates definite difficulties, since h usually is unknown. The most expedient for surmounting these difficulties is to use the minimax approach, considering h in (4.3.8) to be equal to the least value, with which it is still possible to detect the target with given probability.

4.3.3. Case of Slow Fluctuations

In the case of slow fluctuations of the reflected signal ($\tau_{кс} \gg T$ or, the same, $\Delta f_c T \ll 1$), the solution of equation (4.3.5) is constant

$$v(t_1, t_2) = v_0 = \frac{\frac{P_c}{N_0^2}}{1 + \frac{P_c T}{2N_0}}.$$

Here, in accordance with (4.2.5) and (4.3.4)

$$L(y) = v_0 \left| \int_0^T u(t-\tau) e^{i\omega_1 t} y(t)\, dt \right|^2. \qquad (4.3.9)$$

Optimum processing in this case reduces to multiplication of the received signal by the expected $(u(t-\tau)e^{i\omega_1 t})$, integration during the time of observation and formation of the square of the modulus of complex oscillation on output. The block-diagram, carrying out the enumerated operations, is shown in Fig. 4.4. The diagram consists of two quadrature channels, in each of which the signal is mixed with the reference voltage $u_0(t-\tau) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} [\omega_1 t + \psi(t-\tau)]$ and is integrated during the time T. The results of integration are squared, added and are compared with the threshold.

If signal u(t) has a finite duration, then multiplication by the expected signal and integration can be replaced by filtration, considering the factor $u(t-\tau)e^{i\omega_1(t-\tau)}$ in (4.3.9) as the pulse reaction of the filter $h_1(\tau-t)$. Addition of the factor $e^{-i\omega_1\tau}$ does not change the magnitude L(y), since $|e^{i\omega_1\tau}|^2 = 1$, and due to the finite duration of the signal, the beginning of reading of time

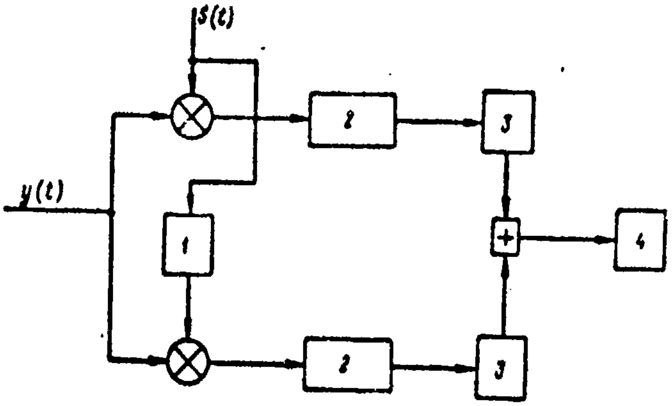

Fig. 4.4. Functional diagram of optimum system of detection for the case of slow fluctuations: 1) phase inverter at 90°; 2) integrator during the time T; 3) square-law generator; 4) relay.

can be selected in such a manner that $u(t) \equiv 0$ at $t > 0$.

Here, the optimum filter is physically realized. Conducting the same reasonings as on the basis of the diagram of Fig. 4.3 for fast fluctuations, we can be convinced that filtration in the considered case also can be carried out on an intermediate frequency, if this frequency is sufficiently high ($f_{\text{пр}} \gg \Delta f_{\text{м}}$, where $\Delta f_{\text{м}}$ — the width of the spectrum of modulation) and the signal of image frequency is suppressed in the input circuits of the receiver.

In those cases, when the distance to the detected target is known, the methods of realization of optimum processing with heterodyning and filtration are, in principle, equivalent. However, if, as this usually occurs, the distance to the target is unknown, then the version with the filter is more preferable, since the output of the filter in various moments of time coincides with the results of optimum processing of signals, corresponding to various distances to the target. In order to separate and compare with the threshold the signal from the target, located at a determined distance d_0, it is necessary to gate the output of the optimum filter in the moment $t = \tau = \frac{2d_0}{c}$ by a sufficiently short pulse.

The received signal y(t) represents the sum of the reflected signal and

noises

$$y(t)=\sqrt{2P_c}E\,\mathrm{Re}\,u(t-\tau)e^{i\omega_1 t+i\varphi}+n(t), \tag{4.3.10}$$

where E and φ are coefficients, considering the chance of amplitude and phase of the signal $(\overline{E^2}=1)$;

n(t) is the noise.

On the output of the optimum filter, we obtain

$$u_\Phi(t)=\sqrt{\frac{P_c}{2}}E\,e^{-i\omega_1 t-i\varphi}\int_{-\infty}^{t}u^*(t_1-\tau)\,u(t_1-t)\,dt_1+$$
$$+\int_{-\infty}^{t}n(t_1)\,u(t_1-t)\,e^{i\omega_1(t_1-t)}dt_1. \tag{4.3.11}$$

As can be seen from this formula, the amplitude of useful signal on the output of the filter is proportional to the function of autocorrelation of the main signal $C(t-\tau,\ 0)=C(t-\tau)$ introduced in Chapter 1. The maximum value of useful component on the output of the filter is obtained at $t=\tau$.

The first component in (4.3.11) may be considered as the reaction of the optimum system, designed for the magnitude of delay of signal t, on the signal with delay τ. i.e., on the signal from a target, not coinciding with the detected target in distance. For good selection in distance it is necessary that the indicated reaction is as small as possible at $t\neq\tau$, i.e., that function $C(x)$ decreases quite quickly with the increase of $|x|$. This function determines, thus, the resolving power of the radar in distance.

For the laws of modulation usually utilized in radar, C(x) represents a rather short pulse (see Section 1.1), the duration of which in a number of cases (for example, for frequency and phase-code modulation) is many times less than the duration of the main signal. In connection with this, the filters, carrying out optimum processing of such signals, frequently are called reducing filters.

Even if the frequency of the signal from the target is different than that expected $(\omega_1+\Delta\omega$ instead of $\omega_1)$. then the amplitude of useful signal on the output of the reducing filter is proportional to $C(t-\tau,\ \Delta\omega)$. The faster

diminishes $C(t-\tau, \Delta\omega)$, with the growth of $|\Delta\omega|$, the higher, obviously, the resolving power ensured by the given main signal according to speed.

Let us consider more specifically, very important for practice, the case of reception of a group of periods of a signal.

The form and duration of the group is determined by the form of the diagram of direction and speed of scanning. We shall consider that the group envelope does not distort modulation of the reflected signal, i.e., or (if the signal is continuous) the duration of the group is great in comparison with the period of modulation, or (if the signal is pulse) the duration of the group is great as compared with the pulse duration. Then (4.3.9) can be rewritten in the form

$$L(\Omega) \sim \left| \sum_{l=-\infty}^{\infty} g(lT_r) \int_{lT_r+\tau}^{lT_r+T_r+\tau} y(t)\,u(t-\tau)\,e^{i\Omega t}\,dt \right|^2, \qquad (4.3.12)$$

where g(t) is the group envelope.

Optimum processing can be ensured by accumulation of the results of processing of separate periods with coefficients $g(lT_r)$. Correlation processing of every period can be replaced by filtration. Pulse reaction of the corrresponding optimum filter is determined by the formula

$$h(t) = \begin{cases} u(-t)\,e^{-i\Omega t}, & 0 < t < T_r, \\ 0 & t < 0,\ t > T_r. \end{cases} \qquad (4.3.13)$$

Thus, every period of modulation is filtered as a separate pulse in duration T_r with intra-pulse modulation u(t). The output signal of the filter is gated in moments $\tau + lT_r$ and the separated short pulses are stored with weight $g(lT_r)e^{i\Omega lT_r}$. If one were to change the beginning of reading of time in such a manner that g(t) = 0 at $t > 0$, then accumulation with weight may be replaced by transmission through the filter with pulse reaction g(—t)cos $\omega_0 t$. Consequently, optimum processing of the signal can be produced in this case with the help of two series filters, carrying out intra- and inter-period processing,

and a square-law detector.

It is possible to reduce optimum operations to the same form with a periodic main signal in the case of fast fluctuations of a reflected signal. Here, the frequency response of the second filter is determined not by the form of the group envelope, but by the spectrum of fluctuations and magnitude of the signal-to-noise ratio in accordance with formula (4.3.8).

4.3.4. Optimum Processing with Arbitrary $\Delta f_c T$

Earlier, we analyzed cases of fast and slow fluctuations of a reflected signal (large and small values of $\Delta f_c T$). By measure of the change of $\Delta f_c T$, the optimum operations should, obviously, change from one of the considered extreme cases to another. In order to trace this change and to definitize the condition of application of the previously used approximations, it is necessary to solve equation (4.2.5) with arbitrary $\Delta f_c T$, at least for certain particular cases. In principle, such a solution can be received for an arbitrary fractional-rational spectral density of fluctuations [60]. Here, the solution of the integral equation reduces to finding Green's function of the linear differential equation with constant coefficients at determined boundary conditions. No principal difficulties are presented by this problem; however, for spectral densities of high order, it is very awkward. Therefore, we, for simplicity, will be limited to consideration of spectral density of the form

$$S_0(\omega) = \frac{1}{1+\left(\frac{\omega}{2\Delta f_c}\right)^2} = \frac{1}{\left|1+\frac{i\omega}{2\Delta f_c}\right|^2}, \qquad (4.3.14)$$

corresponding to the exponential function of the correlation of fluctuations.

Substituting in (4.3.5) the expression for $\rho(t)$ by $S_0(\omega)$ and applying an operator to both parts of the equality

$$1 - \frac{1}{4\Delta f_c^2}\frac{\partial^2}{\partial t_2^2},$$

we obtain

$$v(t_1, t_2) - \frac{1}{4\Delta f_c^2}\frac{\partial^2 v(t_1, t_2)}{\partial t_2^2} + \frac{P_r}{2N_0\Delta f_c}\int_0^T v(t_1, t)\times$$

$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\frac{1}{4\Delta f_c^2}(-i\omega)^2}{1+\frac{\omega^2}{4\Delta f_c^2}}\, e^{i\omega(t-t_2)}d\omega\, dt =$$

$$= \frac{P_r}{N_0^2\Delta f_c}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\frac{1}{4\Delta f_c^2}(-i\omega)^2}{1+\frac{\omega^2}{4\Delta f_c^2}}\, e^{i\omega(t_1-t_2)}d\omega, \tag{4.3.15}$$

Considering that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega = \delta(t),$$

from (4.3.15) we find

$$-\frac{1}{4\Delta f_c^2}\frac{\partial^2 v(t_1, t_2)}{\partial t_2^2} + (1+h)v(t_1, t_2) = \frac{2h}{N_0}\delta(t_1 - t_2). \tag{4.3.16}$$

The solution of differential equation (4.3.16) depends on two constants, determined by boundary conditions for $v(t_1, t_2)$ at $t_2 = 0$ and $t_2 = T$. In order to find $v(t_1, T)$, we apply an operator to equation (4.3.5)

$$1 - \frac{1}{2\Delta f_c}\frac{\partial}{\partial t_2},$$

and then let us assume $t_2 = T$. On the strength of the fact that

$$\left(1 - \frac{1}{2\Delta f_c}\frac{\partial}{\partial t_2}\right)\rho(t_1 - t_2)\big|_{t_2=T} =$$

$$= -[1+\operatorname{sign}(t_1 - t_2)]\,e^{-2\Delta f_c|t_1-t_2|}\big|_{t_2=T} = 0,$$

where

$$\operatorname{sign} t = \begin{cases} 1 & \text{at } t > 0, \\ -1 & \text{at } t < 0, \end{cases}$$

we obtain the first boundary condition

$$v(t_1, T) + \frac{1}{2\Delta f_c}\frac{\partial}{\partial t_2}v(t_1, t_2)\big|_{t_2=T} = 0. \tag{4.3.17}$$

Analogously, at $t_2 = 0$, using operator

$$1+\frac{1}{2\Delta f_c}\frac{\partial}{\partial t_2},$$

we find the second boundary condition

$$v(t_1, 0)-\frac{1}{2\Delta f_c}\frac{\partial}{\partial t_2}v(t_1, t_2)\Big|_{t_2=0}=0. \tag{4.3.18}$$

The same conversions are used for obtaining a differential equation and boundary conditions in the case of spectral densities of a higher order. Here, accordingly, is increased the order of utilized differential operators.

The solution of equation (4.3.16) with boundary conditions (4.3.17), (4.3.18) is recorded in the form

$$v(t_1, t_2)=\frac{2h\Delta f_c}{N_0\sqrt{1+h}}\left\{ e^{-2\Delta f_c\sqrt{1+h}|t_1-t_2|}+ +2\,\frac{h\,\mathrm{ch}\,2\Delta f_c\sqrt{1+h}\,(T-t_1-t_2)+(\sqrt{1+h}+1)^2 e^{-2\Delta f_c\sqrt{1+h}\,T}\,\mathrm{ch}\,2\Delta f_c\sqrt{1+h}\,(t_1-t_2)}{(\sqrt{1+h}+1)^2\left[e^{2\Delta f_c\sqrt{1+h}\,T}-a^2e^{-2\Delta f_c\sqrt{1+h}\,T}\right]}\right\}, \tag{4.3.19}$$

where

$$a=\frac{\sqrt{1+h}-1}{\sqrt{1+h}+1}.$$

In principle, optimum processing of a signal, determined by formulas (4.3.19) and (4.3.4), can be realized in the diagram, given in Fig. 4.1. Filtration in this case, as in that earlier considered, can be carried out on an intermediate frequency. The demodulated signal of intermediate frequency should be passed through a filter with pulse reaction $v(t_1, t) \cos \omega_{\text{пр}} (t_1 - t)$ and move to the phase detector together with the signal from the input of the filter. The output voltage of the phase detector should be integrated during the time of observation T. The filter of the optimum system of signal processing has, as can be seen from (4.3.19), parameters variable in time. In connection with

this, during its technical solution, there can appear a difficulty, significantly greater, than in the realization of a filter, designed for fast fluctuation. Therefore, essential interest is presented by the manifestation of conditions, with which optimum processing, determined by formula (4.3.19), is close (in form or in effectiveness) to one of the earlier considered extreme cases. Here, we are limited to a comparison by form, having left the comparison by effectiveness to the following paragraph, devoted to the investigation of the characteristics of detection.

The first component in (4.3.19) coincides with $v(t_1, t_2)$ during fast fluctuation. In order to explain the distinction of optimum processing at the considered value of $\Delta f_c T$ and processing during fast fluctuation, it is necessary to estimate the order of magnitudes of the remaining components.

The third component diminishes with the growth of $\Delta f_c T$ at any $\Delta f_c T$ not slower, than $e^{-2\sqrt{1+h}\Delta f_c T}$. This decrease occurs faster, the greater the signal-to-noise ratio h, but even at $h \ll 1$, this component may be disregarded, starting from $\Delta f_c T \approx 1$ to 2.

The second component in (4.3.19) slightly depends on $\Delta f_c T$ at t_1, t_2, near to the ends of interval (0, T). However, by measure of removal from the ends, this member also diminishes faster, the greater the $\Delta f_c T \sqrt{1+h}$. To disregard this component means that we allow nonoptimalness of signal processing on some part of the interval of observation. If one were to allow that this part constitutes 20% of the interval of observation, then at $h \ll 1$ it is possible to disregard the considered component in (4.3.19), starting from $\Delta f_c T \approx 10$, and at $h \approx 3$, starting from $\Delta f_c T \approx 5$.

At small $\Delta f_c T$, function $v(t_1, t_2)$ hardly changes in the interval of observation and can be replaced by a constant at values of $\Delta f_c T$, not exceeding in order of magnitude $\frac{0.2 \text{ to } 0.5}{\sqrt{1+h}}$.

The formulated conditions, in the observance of which are near to optimum,

the received above methods of processing fast and slowly fluctuating signals, can be significantly weakened, if one were to compare the considered methods of processing a signal by effectiveness. Such a comparison will be made in the following paragraph.

4.4. Characteristics of Detection of a Signal on A Noise Background

4.4.1. Case of Fast Fluctuations

As already was noted, the general relationships in Section 4.2, with the help of which is determined the form of the characteristics of detection, can be used also when the method of signal processing differs from optimum. In coherent processing of a received signal, these differences, basically, reduce to the difference of the reference signal and frequency response of the filter from the optimum, considered in the preceding paragraph. Here, let us consider the characteristics of detection with arbitrary reference signal and characteristics of the filter. The received results allow to determine not only the effectiveness of the optimum system, but also the degree of lowering of effectiveness, due to the withdrawal from the optimum method of processing.

If processing of signal y(t) is carried out by the system consisting of mixer, filter, square-law detector, accumulator, and relay, then the signal on the input of the relay is determined by the formula

$$L(y) = \frac{1}{N_0} \int_0^T dt \left| \int_0^t h(t - t_1) Z(t_1 - \tau_1) y(t_1) e^{i\omega_1 t_1} dt_1 \right|^2. \qquad (4.4.1)$$

where Z(t) is the reference signal, equal, at optimum processing, to u(t);

h(t) is the pulse reaction of the filter.

Coefficient $\frac{1}{N_0}$ is added for similarity of this expression to the result of optimum processing (4.3.7).

For simplification of notation, we consider the case, when filtration is produced at low frequency. Designating the low-frequency component of the

product $Z(t-\tau_1)y(t)e^{i\omega_0 t}$ by f(t), we can rewrite (4.4.1) [illegible]

$$L(y)=\frac{1}{N_0}\int_0^T\int_0^T v(t_1,t_2)f(t_1)f^*(t_2)\,dt_1dt_2, \tag{4.4.1'}$$

where

$$v(t_1,t_2)=\int_0^T h(t-t_1)h^*(t-t_2)\,dt\approx\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^2e^{i\omega(t_1-t_2)}d\omega. \tag{4.4.2}$$

Upon transition from (4.4.1) to (4.4.1') we replaced the upper limit in the inner integral in (4.4.1) by T, which is allowed due to the proposed physical realization of the filter (h(t) = 0 at t < 0). The last approximate equality in (4.4.2) [illegible] under the condition that $\Delta f_\phi T\gg 1$, where Δf_ϕ — is the effective width of the transmission band of the filter.

The received expression for L(y) coincides with (4.2.5'), and for finding the characteristic function of this magnitude it is possible to use relationships (4.2.6') — (4.2.8'). For the function of correlation $r(t_1, t_2)$ in these relationships, we have

$$\begin{aligned}r(t_1,t_2)=\overline{f(t_1)f^*(t_2)}=\frac{P_c}{2}Z(t_1-\tau_1)u^*(t_1-\tau)Z^*(t_2-\tau_1)\times\\ \times u(t_2-\tau)\rho(t_1-t_2)+N_0|Z(t_2-\tau_1)|^2\delta(t_1-t_2).\end{aligned} \tag{4.4.3}$$

Usually function $v(t_1, t_2)$ changes slowly as compared with the laws of modulation of reference and main signals. Therefore, in (4.4.1'), f(t) may be replaced [illegible] result of averaging this function by time in an interval, sufficient for averaging the modulation, and at the same time sufficiently small, so that [illegible] $v(t_1, t_2)$, and also random amplitudes and phase of reflected signal could, in this interval, be considered constant. Here, the function of correlation $r(t_1, t_2)$ also is averaged in time. Considering Z(t) to be normalized just as u(t), we have

$$\widetilde{r(t_1,t_2)}=\overline{f(t_1)f^*(t_2)}=\frac{P_c}{2N_0}|C_1(\tau-\tau_1)|^2\rho(t_1-t_2)+\delta(t_1-t_2). \tag{4.4.4}$$

where, in the case of a single sending,

$$C_1(\tau_1-\tau)=\frac{1}{T_{\text{эф}}}\int\limits_{-\infty}^{\infty}Z(t-\tau_1)\,u^*(t-\tau)\,dt, \tag{4.4.5}$$

and in the case of a signal of unlimited duration

$$C_1(\tau_1-\tau)=\lim_{T\to\infty}\frac{1}{T}\int\limits_{0}^{T}Z(t-\tau_1)\,u^*(t-\tau)\,dt. \tag{4.4.5'}$$

Substituting (4.4.4) in 4.2.8') and (4.2.7'), we obtain the following equation for function $G(t_1, t_2;\ \lambda)$:

$$G(t_1, t_2;\lambda)-\lambda\int\limits_0^T G(t_1, t;\lambda)\left[v(t, t_2)+\frac{P_c}{2N_0}\left|C_1(\tau_1-\tau)\right|^2\times\right.$$
$$\left.\times\int\limits_0^T v(t, t')\rho(t', t_2)\,dt'\right]dt=v(t_1, t_2)+\frac{P_c}{2N_0}\left|C_1(\tau_1-\tau)\right|^2\times$$
$$\times\int\limits_0^T v(t_1, t')\rho(t', t_2)\,dt'. \tag{4.4.6}$$

For the considered case of fast fluctuations of the reflected signal (assuming that $\Delta f_0 T\gg 1$, and using (4.4.2), we can find the solution of equation (4.4.6), converting both its parts by Fourier $t_1 - t_2$ and disregarding the fringe effects. As the result of substitution of this solution in (4.2.6') we have

$$\Psi(\eta)=\exp\left\{-\frac{T}{2\pi}\int\limits_{-\infty}^{\infty}\ln\left[1-i\eta\left|H(i\omega)\right|^2\times\right.\right.$$
$$\left.\left.\times\left(1+h\left|C_1(\tau-\tau)\right|^2S_0(\omega)\right)\right]d\omega\right\}. \tag{4.4.7}$$

The expression of the characteristic function for the case, when the signal from the target is absent, is determined from (4.4.7) at $P_c=0$. This formula can be used also for determination of the characteristics of the optimum system. Here, $H(i\omega)$ is replaced by the frequency response of the optimum filter (4.3.8).

From (4.4.7), we can easily find the semi-invariants of distribution

p(L)

$$\varkappa_\nu=(\nu-1)!\frac{T}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^{2\nu}[1+h|C_1(\tau-\tau_1)|^2S_0(\omega)]^\nu d\omega. \tag{4.4.8}$$

As it is known [18], $\varkappa_1$ coincides with the mean value of random variable, $\varkappa_2$ -- with dispersion, and relations $\gamma_1=\frac{\varkappa_3}{\varkappa_2^{3/2}}$, $\gamma_2=\frac{\varkappa_4}{\varkappa_2^2}$ are equal to the asymmetry coefficients and excess of the considered distributive law. By measure of increase of T, relations γ_1 and γ_2, as it is easy to see, diminish, which confirms the approximation of the considered distributive law to normal, for which $\gamma_1=\gamma_2=0$. In normal distribution, the probability that magnitude L exceeds the level of operation of relay c (probability of detection), is determined by the formula

$$P\{L>c\}=D=\frac{1}{\sqrt{2\pi\varkappa_2}}\int_c^{\infty}e^{-\frac{(L-\varkappa_1)^2}{2\varkappa_2}}dL=1-\Phi\left(\frac{c-\varkappa_1}{\sqrt{\varkappa_2}}\right), \tag{4.4.9}$$

where by $\Phi(x)$ is designated the integral of probability.

Considering h = 0 in formulas for $\varkappa_1$ and $\varkappa_2$, we can obtain an analogous expression for the probability of false alarm, and excluding from these expressions the magnitude of threshold, we can find the equation of the characteristics of detection

$$D=1-\Phi\left[\sqrt{\frac{\varkappa_{2n}}{\varkappa_2}}\Phi^{-1}(1-F)-\frac{\varkappa_1-\varkappa_{1n}}{\sqrt{\varkappa_2}}\right], \tag{4.4.10}$$

where by $\varkappa_{\nu n}$ is designated the semi-ivariant $\varkappa_\nu$ in the absence of a signal from the target, and by $\Phi^{-1}(p)$ — the function, inverse to $\Phi(x)$.

For a more precise definition of magnitudes of probabilities D and F, corresponding to normal distributive law p(L), it is possible to use the Edgeworth series [46]

$$D=1-\Phi\left(\frac{c-\varkappa_1}{\sqrt{\varkappa_2}}\right)+\frac{\gamma_1}{3!}\Phi^{(3)}\left(\frac{c-\varkappa_1}{\sqrt{\varkappa_2}}\right)-$$
$$-\frac{\gamma_2}{4!}\Phi^{(4)}\left(\frac{c-\varkappa_1}{\sqrt{\varkappa_2}}\right)-\frac{10\gamma_1^2}{6!}\Phi^{(6)}\left(\frac{c-\varkappa_1}{\sqrt{\varkappa_2}}\right)+\ldots \tag{4.4.11}$$

whereby $\Phi^{(i)}(x)$ is designated the ith derivative of the integral of probability.

The considered approximations are correct only at large values of products of $\Delta f_c T$ and $\Delta f_\phi T$, where the less the probability of false alarm and target miss, the slower the probabilities converge to the magnitudes, determined by normal approximation. To receive an expression for distribution, corresponding to the characteristic function (4.4.7), not using the normal approximation and not resorting to specific approximations of frequency response of the filter and spectral density, is impossible. If one were to assume that $S_0(\omega)$, it is possible to approximate a P-shaped curve, having width $2\pi\Delta f_c$, and that the filter is coordinated with the spectrum of fluctuations ($|H(i\omega)|^2$ also will be approximated by a P-shaped curve, having width $2\pi\Delta f_c$), then (4.4.7) will be converted to the form

$$\Psi(\eta)=\frac{1}{[1-i\eta(1+h|C_1(\tau-\tau_1)|^2)]^{\Delta f_c T}}. \tag{4.4.12}$$

With integral $\Delta f_c T$, the distributive law, corresponding to the characteristic function (4.4.12), coincides with the chi-square by distribution with $2\Delta f_c T$ degrees of freedom. Such coincidence is connected with the fact that in the considered case, on the strength of the known theorem of V. A. Kotel'nikov [36], the process on the output of the filter is determined by its discrete meanings, distant from each other by $\frac{1}{\Delta f_c}$, where at right-angle spectral density these values are statistically independent. Furthermore, the narrow-band random process has two independent quadrature components [17], so that the signal in interval (0, T) has $2\Delta f_c T$ statistically independent coordinates, distributed by normal law. As the result of quadratic detection and integration will be formed the sum of squares of the values of all these coordinates, which is subordinated, as is known [46], to chi-square distribution. For probabilities D and F, we obtain the following expressions:

$$D=K_{2\Delta f_c T}\left(\frac{c}{1+h|C(\tau_1-\tau)|^2}\right),$$
$$F=K_{2\Delta f_c T}(c). \tag{4.4.13}$$

where c is the magnitude of threshold, and

$$K_m(c)=\int_c^\infty \frac{x^{\frac{m}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}dx. \tag{4.4.14}$$

The dependence of $K_m(c)$ can be obtained, using the tables of integral law of chi-square of distribution or incomplete gamma-function [61], by which is expressed $K_m(c)$:

$$K_m(c)=1-\frac{\Gamma\left(\frac{m}{2},\frac{c}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}.$$

For facilitating the calculations by the formulas (4.4.13) in Fig. 4.5 and 4.6 are shown graphs of inverse function $c = K_m^{-1}(p)$ for various m. Here, on the axis of ordinates for convenience is the magnitude $\frac{c-m}{\sqrt{2m}}$, remaining finite at $m\to\infty$.

In Fig. 4.5, on the axis of abscissas is log p. This graph is convenient to use for the determination of threshold, corresponding to some value of probability of false alarm. In Fig. 4.6, on the axis of abscissas is log(1 - p), due to which a section of the curve will stretch, where $p\approx 1$.

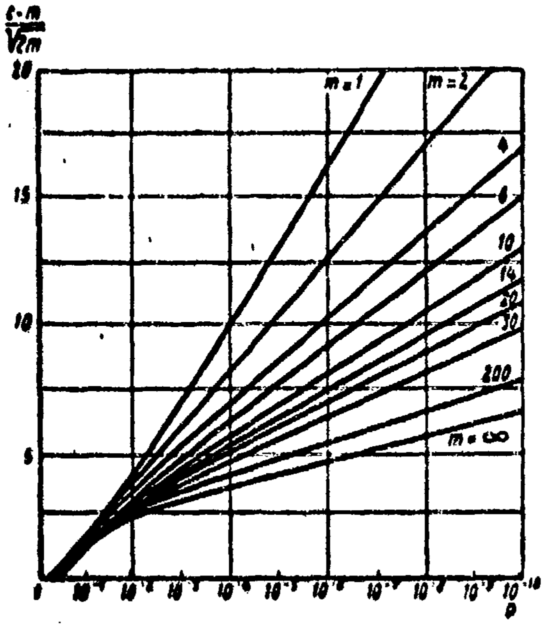

Fig. 4.5. Dependence of $\frac{c-m}{\sqrt{2m}}$ on p at $p \ll 1$.

This graph is convenient for the determination of probability of correct detection or threshold signal-to-noise ratio*, corresponding to the given probability of correct detection. As can be seen from the graphs, at $\Delta f_c T \geqslant 3$ to 5, the change of $\Delta f_c T$ by one leads to a comparatively small change of $K_{2\Delta f_c T}(p)$, so that at fractional values of $\Delta f_c T > 3$, this magnitude can, without appreciable errors, be replaced by the nearest integer and chi-square distribution may be used.

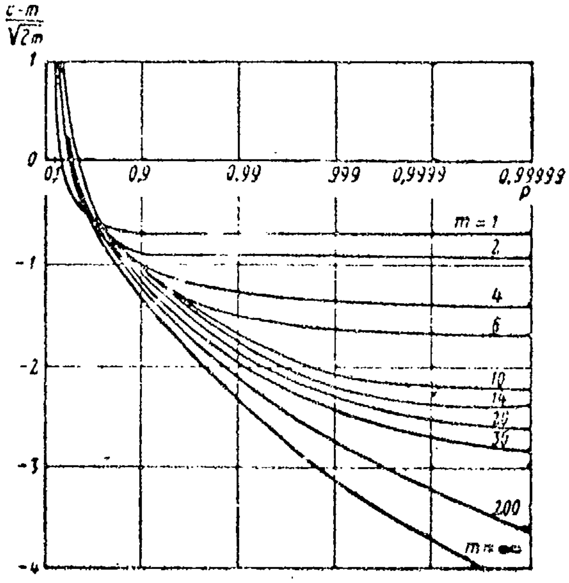

Fig. 4.6. Dependence of $\frac{c-m}{\sqrt{2m_1}}$ on p at $1-p \ll 1$.

Excluding from (4.4.13) the magnitude of threshold c, we obtain the following equation for the characteristics of detection:

$$D = K_{2\Delta f_c T}\left[\frac{K^{-1}_{2\Delta f_c T}(F)}{1 + h_c G_1(\tau_1 - \tau)}\right], \tag{4.4.15}$$

hence, in particular, can be found the magnitude of threshold signal-to-noise ratio, required for guarantee of definite probabilities of F and D. At $\Delta f_c T \gg 1$

$$K^{-1}_{2\Delta f_c T}(p) \approx 2\sqrt{\Delta f_c T}\left[\Phi^{-1}(1-p) + \sqrt{\Delta f_c T}\right]$$

---

*The threshold signal-to-noise ratio originally was called the magnitude of this ratio, with which starts the fast growth of the probability of detection. Gradually, by measure of appearance in theories of detection of strict quantitative methods, in this conception there were somewhat different meanings: The threshold relation signal-to-interference ratio was called the value of this relation, with which is attained a definite probability of correct detection at a given probability of false alarm. In such meaning this term will also be used in this book.

and from (4.4.15) we have

$$h \approx \frac{\Phi^{-1}(1-F)+\Phi^{-1}(D)}{\sqrt{\Delta f_c T}-\Phi^{-1}(D)}. \qquad (4.4.16)$$

The error, allowed in the use of normal approximation, one can determine, by comparing the graphs in Fig. 4.5 and 4.6 at the considered value of m and at $m=\infty$. The received relationships together with the results of Section 4.4.2 will be used later in the analysis of the influence of main factors, determining the threshold value of the signal-to-noise ratio.

4.4.2. Case of Slow Fluctuations. Dependence of Threshold Signal on Width of Spectrum of Fluctuations

The characteristic function $\Psi(\eta)$ for this case also can be received with the help of equation (4.4.6), where at optimum accumulation of signal $v(t_1, t_2) \sim \frac{1}{T}$. Solution of equation (4.4.6) is constant

$$G(t_1, t_2; \lambda) = -\frac{1}{T}\frac{d}{d\lambda}\ln[1-\lambda(1+q_0|C(\tau_1-\tau)|^2)].$$

by substituting in (4.2.6') and producing all necessary calculations, for probabilities of false alarm and correct detection, we have

$$F = e^{-c}, \quad D = e^{-\frac{c}{1+q_0|C_1(\tau-\tau_1)|^2}} = F^{\frac{1}{1+q_0|C_1(\tau-\tau_1)|^2}}, \qquad (4.4.17)$$

where $q_0 = \frac{P_c T}{2N_0} = h\Delta f_c T$ the ratio of energy of the signal to the one-way spectral density of noise (in the case of single sending T is the effective duration of sending, see Section 1.2).

For a slowly fluctuating signal, the use of optimum processing clashes with difficulties, connected with the creation of integrators during the time of observation or narrow-band filters with a band, approximately equal to $\frac{1}{T}$. In practice, the very frequently obtained transmission band of the filter significantly exceeds the coordinated one, which certainly is indicated by the characteristics of detections, which also can be found, proceeding from equation (4.4.6). Without a signal from the target, this equation, as in the case of fast fluctua-

tions, may be solved by a Fourier transform. Obtained as the result of the solution, the expression for the characteristic function should, obviously, coincide with (4.4.7), if h in this formula is equal to zero. Solution of equation (4.4.6) in the presence of a signal from the target can be found in the form of the sum of the solution of the equation without a signal and some function of parameter λ, not depending on t_1, t_2. The corresponding expression for characteristic function at H(0) = 1 has the form

$$\Psi(\eta)= \\ =\frac{1-i\eta}{1-i\eta(1+q_0|C_1(\tau-\tau_1)|^2)}\exp\left\{-\frac{T}{2\pi}\int_{-\infty}^{\infty}\ln(1-i\eta|H(i\omega)|^2)d\omega\right\}. \qquad (4.4.18)$$

At $\Delta f_\Phi T \gg 1$, the distribution for L without a signal from the target can approximately be replaced by normal distribution. Then, if $1-D \ll 1$ and $F \ll 1$, then the equation of characteristics of detection has the form

$$D \approx \exp\left\{-\frac{\Phi^{-1}(1-F)\sqrt{\Delta f_{\Phi 1}T}}{1+q_0|C_1(\tau-\tau_1)|^2}\right\}, \qquad (4.4.19)$$

where

$$\Delta f_{\Phi 1}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^4 d\omega.$$

With not very large $\Delta f_\Phi T$, a convenient approximation for the characteristics of detection can be obtained, if one were to consider the amplitude-frequency response of the filter to be right-angle. Then

$$\Psi(\eta)=\frac{1}{[1-i\eta(1+q_0|C_1(\tau-\tau_1)|^2)](1-i\eta)^{\Delta f_\Phi T-1}}. \qquad (4.4.20)$$

If there is no target ($q_0 = 0$), then the characteristic function (4.4.20) corresponds to chi-square distribution with $2\Delta f_\Phi T$ degrees of freedom. If there is a target, then the distributive law is obtained in the form of convolution of chi-square distribution with $2\Delta f_\Phi T$ degrees of freedom and exponential distribution

$$\frac{1}{1+q_0|C_1(\tau-\tau_1)|^2}\, e^{-\frac{x}{1+q_0|C_1(\tau-\tau_1)|^2}}.$$

Substituting this convolution in the expression for probability of correct detection and producing partial integration, we obtain, at $F \ll 1 - D$, an approximate equation of characteristics of detection in the form

$$D \approx \exp\left\{-\frac{K^{-1}_{2\Delta f_{\Phi}T}(F) - 2(\Delta f_{\Phi}T - 1)}{2(1 + q_0 |C_1(\tau - \tau_1)|^2)}\right\}. \tag{4.4.21}$$

This formula also preserves its meaning at $\Delta f_{\Phi}T \approx 1$, i.e., in the case, when the filter band is coordinated with the time of observation. In this case (4.4.21) changes into (4.4.17). At $\Delta f_{\Phi}T \to \infty$, (4.4.21) changes into (4.4.19).

Obtained in paragraph 4.4.1 and 4.4.2, the formulae allow to investigate the dependence of threshold signal-to-noise ratio on the form of reference signal, the transmission band of the filter and width of the spectrum of fluctuations of the reflected signal. Here we will be limited to the consideration of the latter from the shown dependencies, assuming that the signal is subjected in the receiver to optimum processing $|C_1(\tau - \tau_1)|^2 = 1$, and by approximating the spectral density of fluctuations of the reflected signal by a P-shaped curve. In this case, at $\Delta f_c T \ll 1$, from (4.4.17) we have

$$q_0 = \frac{\ln \frac{1}{F}}{\ln \frac{1}{D}} - 1 \approx \frac{\ln \frac{1}{F}}{\beta}. \tag{4.4.22}$$

The last uality in formula (4.4.22) is correct at $\beta = (1-D) \ll 1$ and $F \ll 1$ (these conditions with practically interesting value of F and β usually are executed). At $\Delta f_c T \gg 1$, the threshold magnitude q_0 is determined from (4.4.15)

$$q_0 = h\Delta f_c T \approx \left[\frac{K^{-1}_{2\Delta f_c T}(F)}{K^{-1}_{2\Delta f_c T}(D)} - 1\right] \Delta f_c T. \tag{4.4.23}$$

Using formulae (4.4.22), (4.4.23), it is possible to construct graphs of the dependence of the threshold meaning of the signal-to-noise ratio q_0 from the product of $\Delta f_c T$ at various F and D.

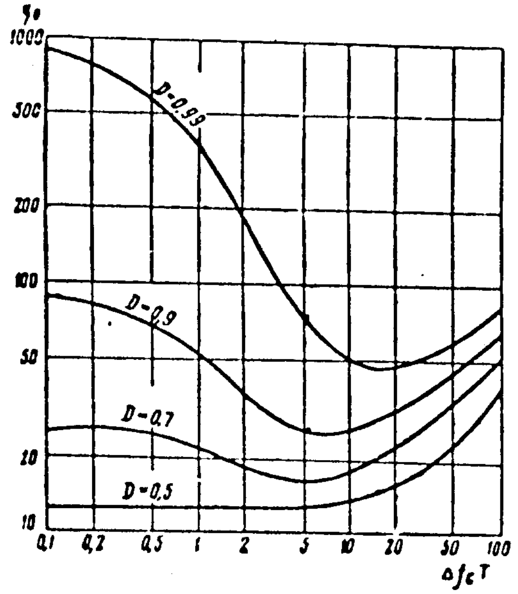

Fig. 4.7. Dependence of threshold signal-to-noise ratio on width of spectrum of signal fluctuations.

Such a dependence is shown in Fig. 4.7 at $F = 10^{-4}$. At $D > 0.5$, the function $q_0(\Delta f_c T)$ has a minimum at $\Delta f_c T \approx 4[\Phi^{-1}(D)]^2 + 1$ which is the result of the influence of two factors, each of which separately leads to a decrease of the threshold signal-to-noise ratio, simultaneously weakening the influence of the other factor. These factors are the coherence of the stored signal and decrease of relative magnitude of fluctuation of the stored signal with the increase of number of statistically stored independent values of reflected signal.

In the construction of graphs in Fig. 4.7, a section of the curves, on which the conditions of the derivation of formulas (4.4.22) and (4.4.23) are not executed, is received by interpolation. An accurate calculation of the characteristics of detection in this section is connected with essential mathematical difficulties. At the same time, the width of this section is small: starting with $\Delta f_c T \approx 3$, for the solution of equations, determining the form of optimum processing and characteristics of detection, Fourier transform can be used (see paragraph 4.3.4). The results, received for right-angle spectral density $S_0(\omega)$, can be used as approximate also for other approximations of spectral

density.

4.4.3. Dependence of Threshold Signal-to-Noise Ratio on the Law of Modulation of Reference Signal and Characteristics of a Reducing Filter

The magnitude of the threshold signal-to-noise ratio, corresponding to given probabilities of miss and false alarm, essentially depends on how much the reference signal of the receiver (pulse reaction of reducing filter) differs from optimum. From the above-obtained formulae, it is clear that the threshold signal-to-noise ratio changes reciprocally to the square of the modulus of the function of mutual correlation of modulations of the reference and reflected signals $C_1(\tau-\tau_1)$. Thus, from the viewpoint of increase of range of radar from the reference signal it is required that for it the function $|C_1(\tau-\tau_1)|^2$ hardly differs from one. Hence, for every given form of modulation one can determine the specific requirements of accuracy of coincidence of parameters of reference and main signals, and with the use of filtration — the accuracy of coincidence of pulse reaction of the filter with the time-converted law of modulation of the main signal. As examples, let us consider the usual pulse signal and continuous signal with frequency modulation according to the law of symmetric saws.

In the first case, we shall consider that the pulse duration of the reference signal differs from the duration of mains pulses, and the pulses themselves will be considered to be rectangular, i.e.,

$$u(t)=\sqrt{\frac{F_s}{\tau_u}}\operatorname{rect}\left(\frac{t}{\tau_u}\right),\quad Z(t)=\sqrt{\frac{F_s}{\tau'_u}}\operatorname{rect}\left(\frac{t}{\tau'_u}\right).$$

At $\tau=\tau_1$, from (4.4.5) we obtain

$$|C_1(\tau-\tau_1)|^2=|C_1(0)|^2=$$
$$=\frac{1}{\tau_u\tau'_u}[\min(\tau_u,\tau'_u)]^2=\min\left(\frac{\tau_u}{\tau'_u},\frac{\tau'_u}{\tau_u}\right).$$

Thus, in the considered case, the threshold signal-to-noise ratio is increased

in proportional to the greater of ratios $\frac{\tau_{\text{н}}}{\tau_{\text{н}}'}$ and $\frac{\tau_{\text{н}}'}{\tau_{\text{н}}}$.

In the case of frequency modulation, let us assume that the deviation of frequency of reference signal $2\omega_m'$ differs from the deviation of frequency of main signal $2\omega_m$. Here, (see Section 1.2)

$$u(t) = e^{i\omega_m t\left(1 - \frac{2|t|}{T_r}\right)}, \quad Z(t) = e^{i\omega_m' t\left(1 - \frac{2|t|}{T_r}\right)}$$

and

$$|C_1(0)|^2 = \\ = \left| \frac{2}{T_r} \int_0^{T_r/2} \cos(\omega_m - \omega_m')\, t \left(1 - \frac{2t}{T_r}\right) dt \right|^2 \approx 1 - 0{,}3\,(\Delta f_{\text{м}} T_r),$$

where $\Delta f_{\text{м}}$ is the difference of maximum deviations of frequencies of reference and main signals.

The last equality in this formula is correct under the condition that $\frac{\pi \Delta f_{\text{м}} T_r}{2} \ll 1$. If one were to assume that a decrease of $|C_1(0)|^2$ by 30% is permissible, then the permissible difference of deviations is determined from the condition $\Delta f_{\text{м}} T_r \approx 1$.

We considered the question of required accuracy of coincidence of the law of modulation of reference signal and pulse reaction of reducing filter with the law of modulation of the main signal. In the case of a signal of the complex-modulated single sending type, the received result is exhausting, and all possible tool errors are described by the function $|C_1(x)|^2$. However, for cases of periodic signal or signal with stationary noise modulation, an essential role is played also by the time stability of the shown characteristics. An accurate calculation of the characteristics of detection, taking into account the instability of the reference signal is a rather awkward problem. Therefore, we will be limited here to an approximate estimate of the influence of these instabilities for the most wide-spread case of periodic signal.

In the case of periodic modulation, the spectrum of useful signal on the output of the mixer (Fig. 4.3) or reducing filter is lined. The width of each

spectral line is determined by the time of observation and the width of the spectrum of fluctuations of the reflected signal. In subsequent narrow-band filtration the lines of this spectrum will be separated, located near the carrier frequency.

The presence of instabilities of the generator of reference signal or reducing filter, leads to additional expansion of the spectral lines and to a decrease of amplitudes of the separated frequency components. The first of these effects can lead to the fact that part of the power will not occur in the narrow-band filter and, consequently, the signal-to-noise ratio will decrease on the output of the filter and the threshold signal-to-noise ratio will be increased. The second effect also leads to impairment of the signal-to-noise ratio on the input of the relay. Its influence may be characterized by introducing some equivalent average level of function $|C_1(\tau-\tau_1)|^2$ in the formulas of paragraph 4.4.1 and 4.4.2. The total increase of the threshold signal-to-noise ratio, due to instability of the reference signal, and noncoincidence of it with the main signal, is determined approximately by the following formula:

$$q_0' \approx q_0 \frac{\Delta f_{\text{н}} + \Delta f_{\text{с}}}{\Delta f_{\text{ф}}} \frac{1}{|C_1(\tau-\tau_1)|^2}, \qquad (4.4.24)$$

where $\Delta f_{\text{н}}$ is expansion of spectral lines due to instability;

$\Delta f_{\text{ф}}$ is the band-width of the filter;

q_0 is the threshold signal-to-noise ratio without calculation of instability.

If $\frac{\Delta f_{\text{н}} + \Delta f_{\text{с}}}{\Delta f_{\text{ф}}} < 1$, then this ratio in formula (4.4.24) is replaced by one.

In those cases, when processing of the received signal is produced with the help of reducing filter, for singling out a signal from a target, at a determined distance, it is necessary to separate the instantaneous value of output voltage of the filter in the appropriate moment of time. This operation is produced with the help of gating of the output of the filter with a narrow gate.

We shall see how the duration of the gate influences the magnitude of the threshold signal. For this purpose, we can use the results of paragraph 4.4.1 and 4.4.2, where reference signal Z(t) with a rectangular gate is recorded in the form

$$Z(t)=\frac{1}{A}\int\limits_{-\frac{\tau_{\text{стр}}}{2}}^{\frac{\tau_{\text{стр}}}{2}} u(t-x)\,dx, \qquad (4.4.25)$$

where

$$A^2=2\mathrm{Re}\,\tau_{\text{стр}}\int\limits_{0}^{\tau_{\text{стр}}}\left(1-\frac{t}{\tau_{\text{стр}}}\right)C(t)\,dt.$$

Substituting Z(t) in expression (4.4.5) for $C_1(\tau-\tau_1)$, we obtain

$$C_1(\tau-\tau_1)=\frac{1}{A}\int\limits_{-\frac{\tau_{\text{стр}}}{2}}^{\frac{\tau_{\text{стр}}}{2}} C(\tau-\tau_1+x)\,dx. \qquad (4.4.26)$$

If the duration of the gate significantly exceeds the width $\Delta\tau$ of the main maximum of function $C(\tau)$, then

$$|C_1(0)|^2\approx\frac{\Delta\tau}{\tau_{\text{стр}}}. \qquad (4.4.27)$$

The relationship obtained here can be used in the working out of requirements for developed generators of reference signals, reducing filters and mechanisms, carrying out gating of output voltage of a filter, and also in the quantitative estimate of the influence of the tool errors inherent to these mechanisms. In the future, for simplification of calculation, we shall consider that these errors are absent.

4.4.4. Dependence of Threshold Signal-to-Noise Ratio on the Width of the Transmission Band of the Filter

As already was noted, usually the transmission band of the filter, carrying out coherent accumulation of the received signal, cannot be made narrow enough for matching with the time observation or (in the case of fast fluctuations of reflected signal) with the width of the spectrum of fluctuations. In connection with this, an essential interest is presented by the question of how the

magnitude of the threshold signal-to-noise ratio is influenced by expansion of the filter band and deviation of form of its frequency response from optimum. It is possible to consider this question also with the help of results of paragraph 4.4.1 and 4.4.2.

We shall start from the case of slow fluctuations of reflected signal. With a rectangular frequency response of the filter, the threshold signal-to-noise ratio is determined from (4.4.21)

$$q_0 \approx \frac{K^{-1}_{2\Delta f_\Phi T}(F) - 2(\Delta f_\Phi T - 1)}{2\ln\frac{1}{D}} - 1. \qquad (4.4.28)$$

For practically interesting practice values of D and F, the relation q_0 is usually great as compared with one, which, in connection with this, may be disregarded. Here, for the relation Γ of threshold values of q_0 in the case of a filter with expanded band and filter with $\Delta f_\Phi \approx \frac{1}{T}$, we obtain

$$\Gamma \approx \frac{K^{-1}_{2\Delta f_\Phi T}(F) - 2(\Delta f_\Phi T - 1)}{2\ln\frac{1}{F}}. \qquad (4.4.29)$$

The dependence of Γ on $\Delta f_\Phi T$ at various F is shown in Fig. 4.8. As can be seen from the figure, Γ is increased with a growth slower, the smaller F. At $\Delta f_\Phi T \gg 1$, it is possible to use an approximate formula for Γ

$$\Gamma \approx \sqrt{\Delta f_\Phi T}\,\frac{\Phi^{-1}(1-F)}{\ln\frac{1}{F}},$$

whereby this formula can be derived without any assumptions about the frequency response form. From the presented relationships, it is clear that the threshold value of q_0 is increased upon expansion of the filter band slowly, so that selection of this parameter of the receiving mechanism is slightly critical.

To analogous conclusions leads the consideration of the dependence of $q_0(\Delta f_\Phi T)$ during fast fluctuation, when distribution of voltage on the input of the relay

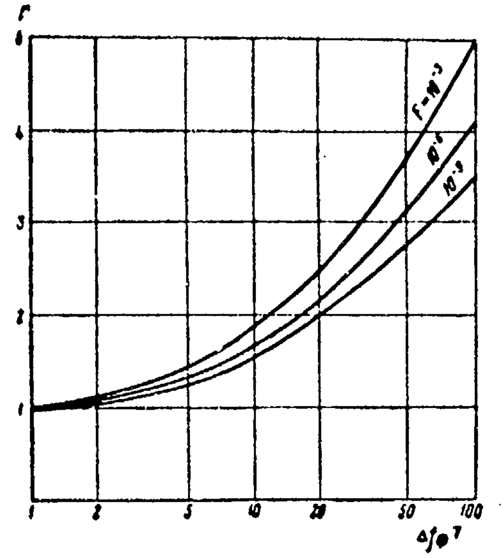

Fig. 4.8. Dependence of relation Γ on the width of the transmission band of the filter for the case of slow fluctuations.

may be considered normal. Considering $\Delta f_c \ll \Delta f_\phi$, from (4.4.8) we obtain

$$\left.\begin{aligned} \varkappa_1 - \varkappa_{1п} &\approx q_0, \\ \varkappa_{2п} &= \frac{T}{2\pi}\int_{-\infty}^{\infty} |H(i\omega)|^4 d\omega = a\Delta f_\phi T, \end{aligned}\right\} \tag{4.4.30}$$

where a is the coefficient of proportionality, depending on the form of frequency response. For the rectangular characteristic of the filter $a = 1$, for LRC-filter $a = 0.5$, for the Gaussian characteristic of the filter $a = 0.7$.

Substituting (4.4.30) in (4.4.10) and considering $\varkappa_2 \approx \varkappa_{2п} (h \ll 1)$, we obtain

$$q_0 \approx \sqrt{a\frac{\Delta f_\phi}{\Delta f_c}}\,[\Phi^{-1}(1-F) + \Phi^{-1}(D)]\sqrt{\Delta f_c T}, \tag{4.4.31}$$

from which it is clear that during fast fluctuation q_0 also grows approximately in proportion to $\sqrt{\Delta f_\phi}$. It is possible to show by means of completely analogous calculations that at $\Delta f_\phi \ll \Delta f_c$ the relation of q_0 changes as $\sqrt{\frac{\Delta f_c}{\Delta f_\phi}}$. With intermediate values of $\frac{\Delta f_\phi}{\Delta f_c}$, the dependence of $q_0(\Delta f_\phi T)$ is obtained more complicated.

As an example, let us consider the case of exponential function of the

correlation $\rho(t) = e^{-2\Delta f_c |t|}$. Here,

$$\varkappa_1 - \varkappa_{1n} = q_0 \frac{x}{1+x},$$

$$\varkappa_{1n} = \frac{\Delta f_c T}{2} x, \quad \varkappa_1 = \frac{\Delta f_c T}{2} x\left[1 + 2h\frac{1+2x}{(1+x)^2} + h^2\frac{(1+x)^2+x}{(1+x)^3}\right],$$

where

$$x = \frac{\Delta f_\Phi}{\Delta f_c}.$$

In Fig. 4.9 is shown the dependence of $q_0(x)$, calculated by the formula (4.4.10) with the use of expressions found for $\varkappa_1$ at D = 0.9, F = 10^{-4}, $\Delta f_c T = 20$. In the same figure, for comparison, the dotted line shows the same dependence, calculated by approximate formula (4.4.31).

From comparison of curves it is clear that the law

$$q_0' \approx q_0 \sqrt{\max\left(\frac{\Delta f_\Phi}{\Delta f_c}, \frac{\Delta f_c}{\Delta f_\Phi}\right)},$$

where q_0 is the magnitude of threshold signal-to-noise ratio with coordinated filter, is observed quite well.

From the received relationships it is clear that selection of the filter band comparatively small influences the quality of detection.

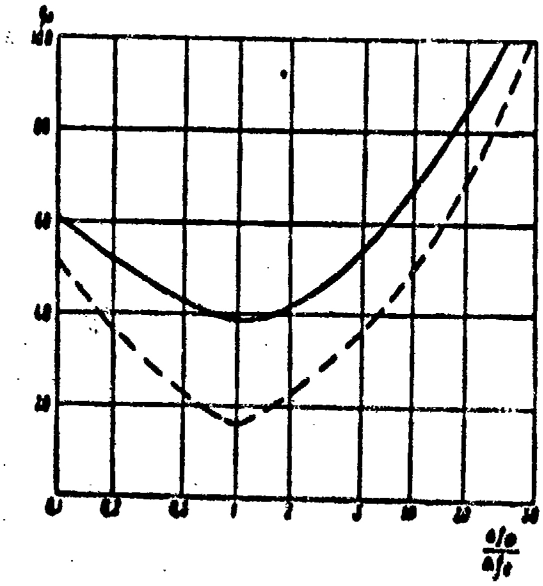

Fig. 4.9. Dependence of threshold signal-to-noise ratio on the width of transmission band of filter for the case of fast fluctuations.

It follows from this that the dependence of the transmission band of the optimum

filter on the signal-to-noise ratio h may not add a large value, in replacing h by its minimum value. In exactly the same way, the form of frequency response of the filter is essentially small. These facts show that the reliability of detection, near to that potentially possible, can be ensured by rather simple technical means.

4.5. Multichannel Systems of Detection

In accordance with the order of the account, outlined in the end of Chapter 3, we recently considered the problem of detection of a target, the distance and speed of which are known.

If, however, the distance to the target and its speed are unknown, then during the time of existence of a signal from targets, located in a given direction, it is necessary to have some method of inspection of all a priori possible distances and frequencies. This is possible to do, either by using a multichannel system, covering the considered range of delays and frequencies, or by carrying out search in distance and speed with the help of one channel, or by using a combination of these methods. The multichannel system, in which is produced a comparison with the threshold of relations of verisimilitude, corresponding to all possible distances and speeds, is equivalent, as is shown under very general conditions in Chapter 3, to an optimum target detection system with unknown parameters. A system with search gives, as shows the later introduced discussions, an essential loss in the free-space ranges as compared with such a multichannel system.

Inasmuch as the total time T_0 of examination of the given direction during the usually utilized uniform survey by angles is limited, the introduction of search is inevitably connected with reduction of time T, expended in examination of every distance and frequency, in a number of times, equal to the product of relations of width of ranges, examined in the process of search by distance and by speed, to the width of ranges, examined simultaneously. Reduction of time T

entails a fast increase of the threshold signal-to-noise ratio: during slow fluctuation $(\Delta f_c T \ll 1)$, the threshold value of power of reflected signal grows with the decrease of T, as $\frac{1}{T}$, and during fast fluctuation - approximately as $\frac{1}{\sqrt{T}}$. In actual conditions, the number of examined elements of solution by distance and speed can reach many thousand. Therefore, the use of multichannel processing in the contemporary radar sets is inevitable. Optimum and quasioptimum methods of construction of one channel of detection were considered above. Here we will be occupied with questions of combination of separate channels in a multichannel system and the possible methods of simplification of these systems.

With the multichannel processing of a signal, the number of utilized channels is determined by the width of examined ranges of Doppler frequencies and delays, and also the permissible detuning of channels by these parameters. This permissible detuning, in turn, depends on the interval of distance, covered by every channel separately, i.e., on the resolving power of the radar by these parameters.

Let us consider this dependence for the case of channels, detuned by speed (Doppler frequency). The channels in this case are narrow-band filters detuned in frequency. The level, on which frequency responses of the filters are covered, should be selected, proceeding from the permissible decrease of probability of detection of the signal, the frequency of which is found on the joint between channels. The probability D_2 of exceeding the level of operation at least in one of the channels is somewhat larger, than the probability of exceeding the threshold in one channel with the same value of detuning of frequency of the signal relative to average frequency of the filter. This increase occurs due to the incomplete correlation of noises in neighboring filters. To produce calculation of probability D_2 in the general case, taking into account signal fluctuations during the time of observation and possible post-defector accumulation is impossible. Let us consider a more particular case, when the output of

the detector is directly compared with the threshold, and the signal during the time of observation is not fluctuating. In the presence of post-detector accumulation and fluctuations, the character of the dependencies, apparently, will not change.

Probability D_2 in the considered case is recorded in the form

$$D_2 = 1 - \int_0^c \int_0^c p(r_1, r_2)\, dr_1\, dr_2, \tag{4.5.1}$$

where $p(r_1, r_2)$ is the joint distribution of envelope squares in channels;

$\underline{c}$ is the threshold of operation.

Joint distribution of envelope squares has the form (2.4.44)

$$p(r_1, r_2) = \frac{1}{R_{11}^2 (1-\rho^2)} e^{-\frac{r_1 + r_2}{R_{11}(1-\rho^2)}} I_0\left(\frac{2\rho \sqrt{r_1 r_2}}{R_{11}(1-\rho^2)}\right), \tag{4.5.2}$$

where

$$R_{11} = 1 + q_1 \left| H\left(i \frac{\Delta\omega_1}{2}\right)\right|^2;$$

$$\rho = \frac{1}{R_{11}} \left| \frac{1}{2\pi \Delta f_\Phi} \int_{-\infty}^{\infty} H(i\omega) H^*(i\omega - i\Delta\omega_1)\, d\omega + \right.$$

$$\left. + q_1 H^2\left(\frac{i\Delta\omega_1}{2}\right)\right|; \tag{4.5.4}$$

$q_1 = \frac{P_c}{2N_0 \Delta f_\Phi}$ is the signal-to-noise ratio in the filter band Δf_Φ;

$H(i\omega)$ is the frequency response of the filter;

$\Delta\omega_1$ is detuning.

Formula (4.5.4) may be simplified, if one were to consider the phase response of the filter within limits of the transmission band to be linear. Then $H(\omega)$ is replaced by $|H(\omega)|$. For calculation (4.5.1), it is convenient to use decomposition (4.5.2) by Laguerre polynomials [17]. As a result, we receive

$$D_2 = 2e^{-\frac{c}{R_{11}}} - e^{-2\frac{c}{R_{11}}} - \sum_{k=1}^{\infty} \frac{\rho^{2k}}{(k!)^2} \left[\frac{d^{k-1}}{dx^{k-1}} x^k e^{-x}\right]^2_{x = \frac{c}{R_{11}}} \tag{4.5.5}$$

At any $0 \leqslant \rho \leqslant 1$, the series in (4.5.5) converges rather quickly and can be used for practical calculations. In Fig. 4.10 is the dependence of D_2 on

$\Delta f_1 = \frac{\Delta\omega_1}{2\pi}$ for $D_1 = D_2(0) = 0.9$ with two approximations of frequency response of the filter: Gaussian (solid curve) and in the form of two series connected RLC-filters (dotted curve). The considered dependence, as is simple to be convinced of, hardly differs from that calculated upon disregarding the weakening of the dependence between noises in channels detuned in frequency. This is connected with the fact that at $1 - D_2 \ll 1$, the relation $q_1 \gg 1$ and with the change of correlation of noises in channels [first component in (4.5.4)] ρ changes little. Due to this, the decrease of probability of detection of a target on the joint of the two filters practically completely is determined by the level of intersection of frequency responses. The threshold signal is increased by approximately $\frac{1}{\left|H\left(\frac{i\Delta\omega_1}{2}\right)\right|^2}$ times.

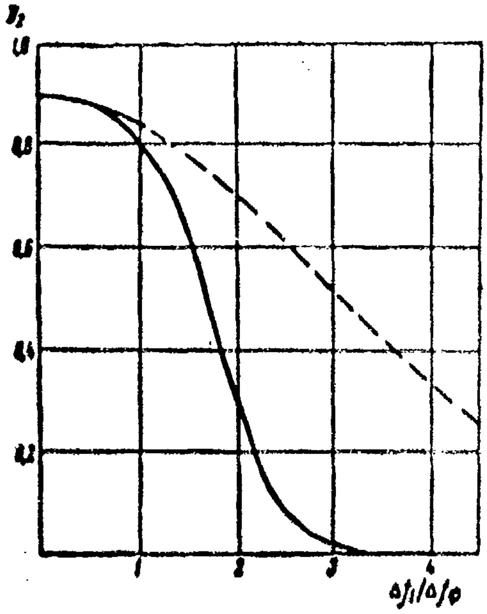

Fig. 4.10. Dependence of probability of correct detection on detuning of channels by frequency.

By designing a permissible decrease of probability of correct detection of target on the joint between channels, one can determine the permissible magnitude of detuning and the number of channels necessary for covering the given range.

In a similar manner is solved the problem of selection of a number of channels in distance. Here, $H(i\omega)$ in the received above formulas is replaced

by $C_1(\tau)$. We can also formulate and solve an inverse problem: on the selection of the range covered by one channel, with a given width of a priori interval and given number of channels. Such a selection can be made, for example, proceeding from the requirement of maximum probability of detection, averaged by a priori interval. However, the practical value of such a problem is comparatively small.

In radar sets, possessing resolving power in distance and speed, the use of multichannel processing in a clear form, when each pair of values of distance and speed, selected in accordance with permissible detuning, corresponds to a separate channel (Fig. 4.2--4.4), connected with excessive complication of the receiving mechanism. Therefore, a great practical interest has the search for methods of simplification of a system of detection by means of unification of some part of the functional elements of various channels, the change of the law of modulation in the process of work of the radar, etc. Of course, the question of the use of that or another method of construction should be solved in every specific case separately, taking into account the specific requirements of the given radar set and state of technology at the time of its development. Here will be considered only the principal possibilities of reduction of the number of channels and combination of their elements.

First of all, simplification is possible due to replacement of the correlated method of processing by filtration. Here, one reducing filter can be used for all distances and for the interval of frequencies, in which function $|C(0, \Omega)|^2$ hardly differs from one. In the majority of practical cases a periodic signal is used. Here, one intra-period reducing filter can be used for all distances and all frequencies, for which $|C_0(0, \Omega)|^2$ is close to one

In practice, the width in frequency of the main maximum of this function of indefiniteness usually considerably exceeds the width of the a priori interval of Doppler frequencies. In this case, one reducing filter can be used for processing all expected signals. A corresponding block-diagram of a multichanne

system of detection has the form, shown in Fig. 4.11. The signal from the output of the reducing filter by means of gating is distributed between channels in distance, in each of which is a block of filters, carrying out selection of targets by speed. If one shortening filter does not cover the entire range of Doppler frequencies, it is possible to use the totality of several systems of such form.

Another possibility of combination of functional elements of separate channels with a periodic signal is connected with the use of storing mechanisms of the potentialoscope or delay line type with feedback.

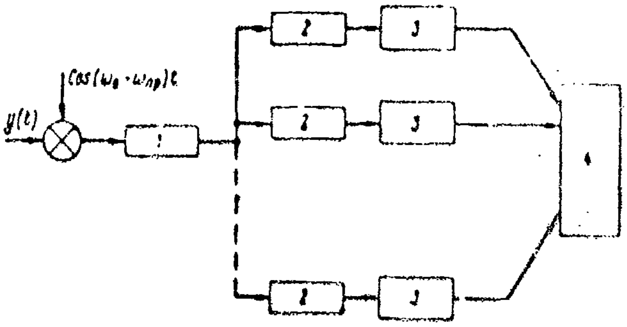

Fig. 4.11. Functional diagram of multichannel system of detection with reducing filter: 1) reducing filter; 2) gating amplifier; 3) block of filters, detectors and integrators (in the case of fast fluctuation); 4) relay block.

Here, the signal from the output of the reducing filter (or totality of reducing filters, if the range of Doppler frequencies is very wide) is mixed with signals from heterodynes, frequency-detuned, whereby on each frequency will separated two quadrature components (Fig. 4.12). The received sequences of low-frequency pulses are coherently stored after which, summation of the squares of quadrature components will form an envelope square. Then, if this is necessary, incoherent accumulation of the signal can be produced. Output voltage of every channel is fed to the relay. The moment of actuation of the relay indicates the magnitude of delay of the signal, and the number of the channel- the magnitude of Doppler shift. If coherent accumulation of the signal for all distances is simultaneously done at intermediate frequency (for example, with the help of ultrasonic delay line with feedback), then the number of channels decreases twice.

The square of the envelope in this case is obtained by detection of oscillations on the output of the accumulator.

In certain cases, as, for example, in phase-code manipulation, the reducing filter can be made at low frequency.

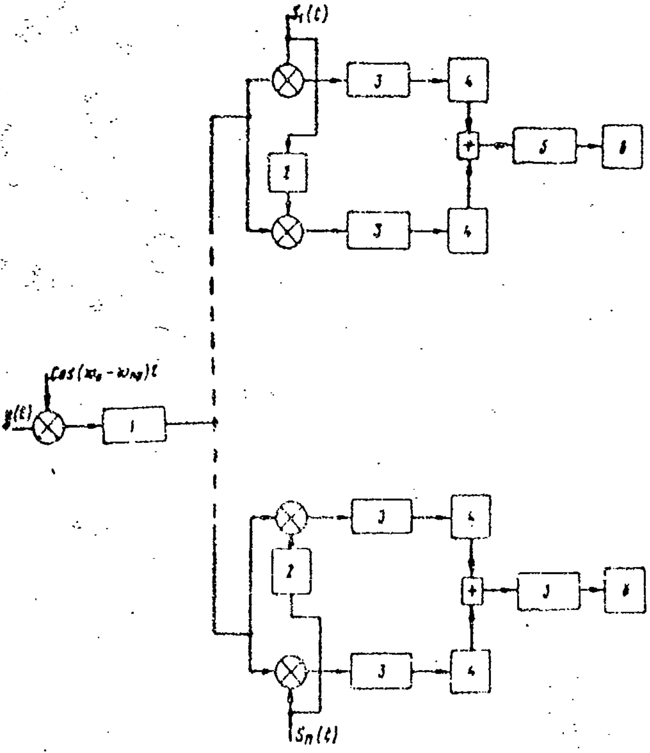

Fig. 4.12. Functional diagram of multichannel system of detection with reducing filter and accumulators of the potentialoscope type: 1) reducing filter; 2) phase inverter at 90°; 3) storing mechanism; 4) square-law generator; 5) storing mechanism; 6) relay.

Very radical means of reduction of the number of frequency channels is the change of the law of modulation in the process of work of the radar. This method is useful, apparently, only for radar sets of special assignment, in problems of which does not enter continuous observation of the newly appearing targets. Such are, in particular, radar sets, passing in conditions of detection after target lock-on.

For these radar sets it is possible, for example, in scanning conditions to use continuous emission. Here there is resolving power only by speed, realized in the receiving mechanism with the help of block of filters detuned

relative to one another, covering the a priori interval of Doppler frequencies. After lock-on of the assumed target (or targets) by speed there occurs the connection of additional modulation, ensuring the required resolving power by distance, and to the frequency of the filter, seizing the target, is tuned the corresponding number of channels by distance. Losing, in the use of such a system, time approximately by two times, we considerably diminish the number of receiving channels.

If in the multichannel system without use of filtration the number of channels is equal to mn ($m = \frac{\Delta f}{\Delta f_1}$ is the number of channels by speed, $n = \frac{\Delta \tau}{\Delta \tau_1}$ is the number of channels by distance), then in the considered system this number is equal to in + m (i is the number of channels, in which there occured operation in conditions of continuous emission). As compared with the system with reducing filters and accumulators, we obtain a $\frac{m}{i}$ - multiple gain in the number of these mechanisms.

The considered method of change of the law of modulation can be subjected to various modifications. Its main idea consists of the fact that the resolving power by distance is increased after lock-on by speed, due to which is attained economy in a number of channels. The increase of resolving power can be smooth. In this case, separation of locked-on targets by distance occurs in the process of detection. Losses in time, occuring with the change of the law of modulation, usually are immaterial, since an increase of resolving power occurs after lock-on by angle, when scanning by angles stops. It is necessary, however, to consider that at a sharp increase of resolving power, the signal of the target at a certain time is lost (until there occurs capture by distance). This time should be quite small, so that the Doppler frequency of the target does not go beyond the limits of the discriminator band and that the target does not exceed the limits of the diagram of directivity.

Let us see how the probabilities of correct detection and false alarm for

the considered system are connected during intermittent change of modulation with probabilities F and D at each stage and with the average frequency of false alarm. If the permissible frequency of false alarm is equal to $f_{\text{тр}}$ and the time expended for detection of the target, is equal to T_0, then the probabilities F' and F'' of false alarm on the first and second stages must satisfy the relationship

$$f_{\text{тр}}T_0 = [1-(1-F)^m][1-(1-F'')^n] \approx mnF'F''. \qquad (4.5.6)$$

The probability of correct detection D is equal to the product of probabilities D' and D" of capture by speed and by distance, if the time of observation is great as compared to the time of correlation of fluctuations of the reflected signal. During slow fluctuation, for the determination of D is required a special calculation. If $1 - D' \ll 1$ and $1 - D'' \ll 1$, the probability of D can be considered approximately equal to the smaller of probabilities D' and D".

The following stage of simplification of the receiving mechanism is abandonment of the high resolving power by speed, ensured in the use of a periodic signal, and transition to incoherent processing. This form of processing gives, however, an essential loss in the threshold signal (see Chapter 5).

In conclusion let us consider still another question, connected with an estimate of the parameters of the detected target in conditions of detection. As was noted in Chapter 3, with uniform a priori distribution, the optimum method of estimate consists in the comparison of the logarithms of relations of verisimilitude, occuring on the output of the channels of detection. As an estimated value, the parameter is selected then, for which the relation of verisimilitude is maximum (principle of maximum of verisimilitude).

With low probabilities of false alarm, exceeding the threshold of operation with great probability occurs only in that channel, where the signal is.

It is natural to take the values of the parameters corresponding to this channel as estimated ones. For comparison of this method of estimation with the maximum verisimilitude method, it is necessary to calculate the corresponding probabilities of errors. We will conduct the calculation of probabilities for the case of slow fluctuations of reflected signal and optimum processing and use of the narrow-band filter, coordinated with the time of observation. Here, the voltage on the output of the channels is distributed according to exponential law (4.4.17). If these voltages are independent, then the probability of detection of target and correct estimate of its parameter with the use of the maximum verisimilitude method is recorded in the form

$$\frac{1}{1+q_0}\int_{c}^{\infty} e^{-\frac{x}{1+q_0}}\,dx\left(\int_0^x e^{-z}\,dz\right)^{m-1} = D\sum_{l=1}^{m-1}(-1)^l\binom{m-1}{l}\frac{F^l}{1+l(1+q_0)} \approx D\left(1-(m-1)\frac{F}{2+q_0}\right), \qquad (4.5.7)$$

where m is the number of channels.

When the estimate is produced simultaneously with the detection by number of the processing channel, the same probability is equal to

$$D(1-F)^{m-1} \approx D[1-(m-1)F]. \qquad (4.5.8)$$

Inasmuch as mF (the probability of false alarm for the entire multichannel system) is usually small, the difference between probabilities (4.5.7) and (4.5.8) may be completely disregarded, if the probability of miss $\beta \gg mF$, which usually takes place. Thus, the compared methods of estimation practically are equivalent. As already was noted in Chapter 3, such a result of comparison is, from qualitative considerations, quite evident.

4. 6. Detection of Target with Multifrequency Emission

At present in radar very wide distribution was given to the idea of the use, for various targets, of a signal, consisting of oscillations of several (usually

two) carrier frequencies with identical or different laws of modulation [62, 63]. Here we consider quantitatively the effectiveness of use of a multifrequency signal from the viewpoint of increase of the range of the radar. The fact is that with a sufficiently large separation of frequencies, the corresponding maxima of diagrams of secondary emission of the target at various frequencies are displaced relative to one another, due to which the dissection of the sum diagram of secondary emission and relative magnitude of fluctuations of the reflected sign is decreased. The biggest weakening of fluctuations occurs with the statistical independence of reflected signals, corresponding to various carrier frequencies. As it was shown in Chapter 1, the condition of independence of two signals is the smallness of wave lengths, corresponding to the separation frequency, as compared with the dimensions of the target and uncovering of the spectra of modulation of these signals. We shall consider these conditions to be carried out.

For statistically independent signals, the logarithm of the relation of verisimilitude (Section 4.2) is equal to the sum of logarithms of relations of verisimilitude for separate signals. In accordance with this, a circuit of optimum processing of a multifrequency signal is the totality of circuits for separate signals. Voltages on the output of these circuits are summarized and are compared with the threshold. The characteristic function of total voltage on the input of the relay is equal to the product of characteristic functions of the components. Using this fact, it is simple, in principle, to find the characteristics of detection for such a multifrequency system.

Let us consider the case of slow fluctuations. If the signal is processed in optimum form, then the characteristic function $\Psi_s(q)$ is determined by the equality (see Section 4.4)

$$\Psi_s(q) = \prod_{j=1}^{m} \frac{1}{[illegible]} \qquad (4.6.1)$$

where m is the number of utilized frequencies.

At $q_{0j} = 0$, this characteristic function corresponds to the chi-squre distribution with 2m degrees of freedom. Producing inverse Fourier transformation for arbitraries q_{0j}, integrating from c to ∞, and replacing c by $K_{2m}^{-1}(F)$, we obtain an equation of characteristics of detection in the form

$$D = \sum_{j=1}^{m} \frac{\exp\left\{-\frac{K_{2m}^{-1}(F)}{2(1+q_{0j})}\right\}}{\prod_{l=1}^{m}{}' \left(1 - \frac{1+q_{0l}}{1+q_{0j}}\right)}, \tag{4.6.2}$$

where the dash at the sign of the product means that in it does not enter a member with l = j.

The greatest interest is presented by the question of selection of the number of frequency channels, ensuring maximum freespace range at a given total emissive power. Considering that the power is distributed between channels equally $\left(q_{01} = q_{02} = \ldots = \frac{q_0}{m}\right)$, from (4.6.2) we obtain

$$D = K_{2m}\left(\frac{K_{2m}^{-1}(F)}{1 + q_0/m}\right). \tag{4.6.3}$$

This formula coincides in form with (4.4.15) with the only difference that $\Delta f_c T$ is replaced by m. The graph of dependence of q_0 (m) is shown in Fig. 4.13.

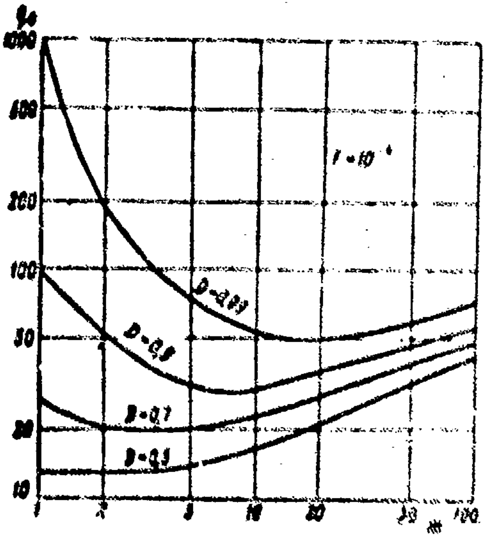

Fig. 4.13. Dependence of threshold signal-to-noise ratio on the number of utilized carrier frequencies.

In its character, this dependence coincides with the dependence of $q_0(\Delta f_c T)$ in Fig. 4.7, which is fully intelligible, since in both cases we mean the dependence of threshold signal-to-noise ratio on the number of statistically independent components of a signal, quadratically summarized in the process of treatment.

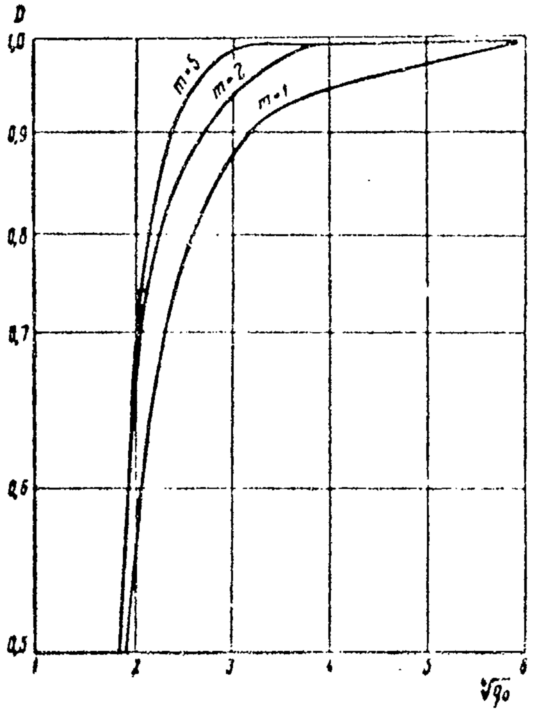

Fig. 4.14. Dependence of probability of correct detection on distance.

As can be seen from the figure, the curves q_0 (m) at D > 0.5 have their minimum at $m = m_{\text{опт}}(D)$, the depth of which is increased with the increase of probability of correct detection. Accordingly, the gain is increased in the threshold signal-to-noise ratio (and consequently, in the free-space ranges) at a fixed number of frequency channels as compared to the case of single-frequency work. Due to this, during multifrequency work at $m = m_{\text{опт}}$, the probability of detection is more quickly increased with the decrease of distance (Fig. 4.14).

It is interesting to note that the curve of dependence q_0 (m) very quickly drops at small m. Due to this, a gain, near to maximum, can be obtained at a comparatively small number of working frequencies (2—4).

During fast fluctuations of a signal reflected from the target, the

characteristics of detection can be calculated by the formulas (4.1.10) and (4.4.11), where the semi-invariants in these formulas are equal to the sums of the semi-ivariants for separate signals. With equal distribution of power between frequency channels and identical spectra of fluctuations at various frequencies, an increase of the number of channels is equivalent to an increase of m times of the time of observation upon simultaneous decrease of the signal-to-noise ratio h. In accordance with this, for the calculation of the characteristics of detection in the considered case can be used, in the appropriate way, the transformed results of Section 4.4. In particular, for rectangular spectral density of fluctuations, using (4.4.15), we obtain

$$q_0 = m\Delta f_c T \left[\frac{K^{-1}_{2m\Delta f_c T}(F)}{K^{-1}_{2m\Delta f_c T}(D)} - 1 \right] = m\Delta f_c T \, \frac{\Phi^{-1}(1-F) + \Phi^{-1}(D)}{\sqrt{m\Delta f_c T} - \Phi^{-1}(D)}. \qquad (4.6.4)$$

The last equality in (4.6.4) is correct at $m\Delta f_c T \gg 1$. From (4.6.4) it is clear that during fast fluctuations of reflected signal, the threshold signal-to-noise ratiois increased with the increase of the number of frequency channels approximately as $\sqrt{m}$. Qualitatively, this is explained by the fact that in this case already there is a sufficient number on the (order of $\Delta f_c T$) of statistically independent components of the signal and further division of power between these components lowers the effectiveness of coherent accumulation of the signal. Thus, during fast fluctuations of reflected signal $(\Delta f_c T \gg 1)$, the use of multifrequency operation from the viewpoint of requirement of increase of range of radar is inexpedient.

We cannot trace the accurate dependence of the ratio of threshold values of q_0 during multifrequency and single-frequency work on $\Delta f_c T$, unfortunately. The assumed character of this dependence is shown by the dotted line for m = 2 and m = 3 at D = 0.9, F = 10^{-4} in Fig. 4.15. By measure of increase of $\Delta f_c T$,

the gain, due to multifrequency, decreases, changing, at some $\Delta f_c T$, into a loss, tending gradually to $\sqrt{m}$.

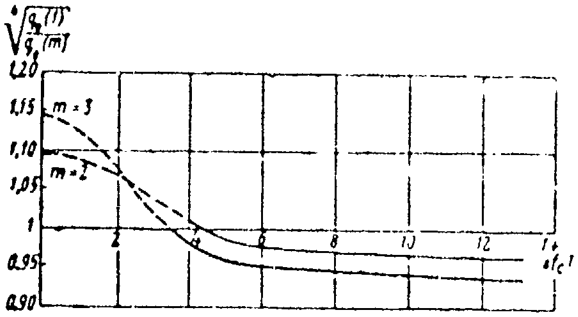

Fig. 4.15. Dependence of gain in range (due to increase in number of frequencies) on the width of the spectrum of signal fluctuations.

4.7. Some Problems of Optimization of Scanning and Searching

In Sections 3.8 and 3.9 we considered some problems of optimization of scanning and searching, whereby their solutions were reduced to the stage, on which it is necessary to use specific characteristics of detection. Now we will continue the solution of problems for the case of a coherent signal. Here, we shall consider only those problems, the solution of which is not connected with large mathematical and purely computational difficulties. Therefore, the account of these questions in this paragraph can be considered as the totality of examples, illustrating the methods of solution of problems, considered in Section 3.8 and confirming the perspectivity of various methods of increase of reliability of detection of a target by means of optimization of scanning and searching. However, the presented solutions also have an independent practical meaning.

We shall start first of all from the problem of the best distribution of time T_0, expended for detection, between cycles during uniform scanning. If ψ is the width of a sector of scanning, and $\Delta\psi$ is the width of the diagram of directivity, then the signal from each target exists for the duration of time

$$T_1 = T_0 \frac{\Delta\psi}{\psi}.$$

which should be distributed between cycles. The meaning of such distribution consists in the decrease of relative level of signal fluctuations with distribution of its energy between several statistically independent components (signals, taken in various cycles).

It is obvious that at $\Delta f_c T_1 \leqslant 1$, the distribution of energy between cycles does not lead to an increase of the free-space range and, from this point of view, subdivision on cycles does not have meaning. Analogously, if in the process of subdivision, the time between two subsequent passages of the beam of the antenna through the target (duration of cycle) is less than the time of correlation of fluctuations, then further subdivision is inexpedient to produce, since smoothing of fluctuations no longer occurs, and the quality of detection is lowered due to the decrease of time intervals, in which the signal is processed coherently. In connection with this, we shall consider only that case, when signals in the cycles are statistically independent. Here, the probability of false alarm F_0 and miss of target β_0 during the time of T_1 are determined by the relationships

$$F_0 \approx mF_1, \quad \beta_0 = \beta_1^m. \qquad (4.7.1)$$

where m is the number of cycles;

F_1 and β_1 are probabilities of false alarm and miss after one cycle.

Let us consider the case, when the duration of reflected signal in each cycle is small as compared with the time of correlation of fluctuations. Here, substituting F_1 and β_1 determined from (4.7.1), in (4.4.22) and considering that the signal-to-noise ratio after one cycle is connected with the signal-to-noise ratio q_0 during the time of T_1 by the formula $q_0 = q\,m$, we obtain

$$q_0 = m\left(\frac{\ln\frac{m}{F_0}}{\ln\frac{1}{1-\sqrt[m]{\beta_0}}} - 1\right). \qquad (4.7.2)$$

The dependence $q_0(m)$ at $F_0 = 10^{-6}$ and various β_0 is shown in Fig. 4.16. In its

character, as one should have expected, this dependence is similar to the dependence q_0 (m) in Fig. 4.13 for the case of incoherent summation of statistically independent components (this dependence was considered in connection with multifrequency work). From the graph it is easy to perceive that the optimum number of cycles corresponds to the probability of omission per cycle, near 0.5. This result can also be obtained by proceeding from (4.7.2). Thus, the optimum number of cycles may be calculated by an approximate formula

$$m_0 = 0{,}3 / \ln \frac{1}{\beta_0}.$$

From a comparison of the curves in Fig. 4.16 and Fig. 4.13 it is clear that an independent comparison with the threshold of results of processing the signal in every cycle gives a loss in distance as compared with incoherent summation of these signals, increased with the increase of m and decrease of β_0. For example, at $\beta_0 = 0.1$ and m = 8, the relative increase of q_0 constitutes 80%.

An analogous calculation can be conducted for the case, when the transmission band of the filter is not coordinated with the time of observation. Here, by substituting (4.7.1) in (4.4.28), we obtain

$$q_0 = m\left[\frac{K^{-1}_{2\Delta f_\phi T_1/m}\left(\frac{F_0}{m}\right) - 2\left(\frac{\Delta f_\phi T_1}{m} - 1\right)}{2\ln\frac{1}{1-\sqrt[m]{\beta_0}}}\right] \approx$$

$$\approx \sqrt{\Delta f_\phi T_1 m}\,\frac{\Phi^{-1}\left(1 - \frac{F_0}{m}\right)}{\ln\frac{1}{1-\sqrt[m]{\beta_0}}}. \qquad (4.7.3)$$

The approximate equality in (4.7.3) is correct at $\Delta f_\phi T_1 \frac{1}{m} \gg 1$. The dependence $q_0(m)$ at $F_0 = 10^{-6}$, $\Delta f_\phi T_1 = 10$ and various β is shown in Fig. 4.17 and has the same character as that considered above. The optimum number of cycles m_0 in this case is somewhat increased; however, due to the weak dependence of q_0 on m in the vicinity of m_0 this increase in practical calculations may be disregarded.

During fast fluctuations of reflected signal, when $\Delta f_c T \gg 1$, the distribution of energy of the signal between statistically independent components even in optimum processing does not lead [see (4.6.4)] to an increase of the

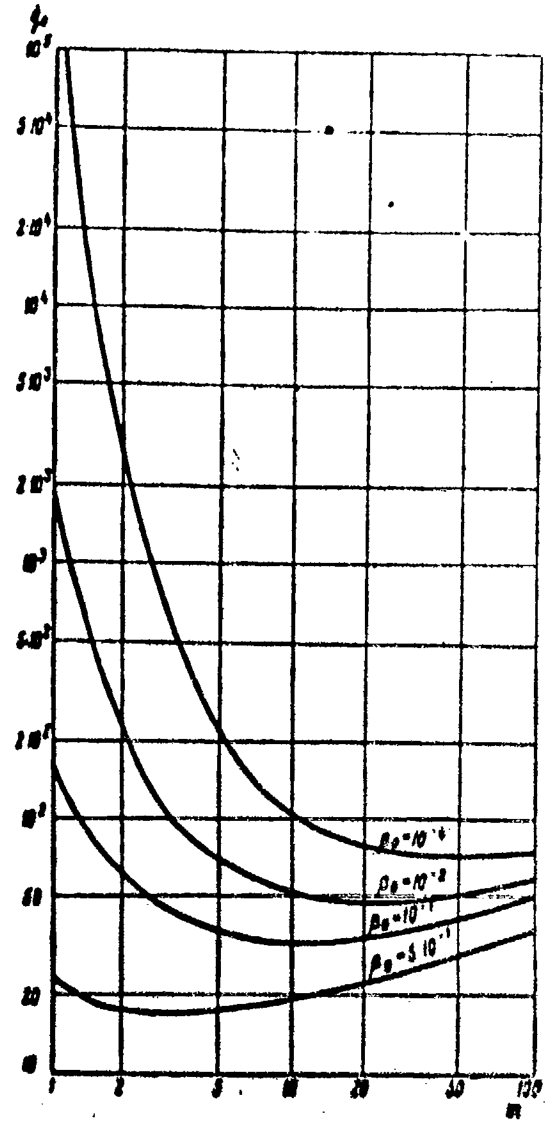

Fig. 4.16. Dependence of threshold signal-to-noise ratio on the number of independent cycles of scanning.

range of the radar. The magnitude of the range loss connected with such distribution is simple in each specific case to calculate by the formulas in paragraph 4.4.1 with the use of (4.7.1).

We considered the problem of effectiveness of division of time expended for detection into statistically independent cycles under the condition that the solution concerning the presence of a target is accepted upon increasing the threshold of operation of the relay even if in one cycle. As was noted in Section 3.8, it is possible to imagine a system, in which the solution on the presence of a target is taken, if in n cycles will occur at least k exceedings of threshold, and to find the optimum number k. The probabilities of correct detection and false alarm are determined in this case by formulas (3.8.2). By substituting in these formulas the characteristics of detection, it is possible to trace the dependence of threshold signal on k and n for any practically interesting case.

In Fig. 4.18 is presented the dependence of q_0 on k at $n = 10$, $F = 10^{-6}$ and various β, constructed for the case of slow fluctuations of a reflected signal, when subdivision into statistically independent cycles is the most effective. Dependence $q_0(k)$ has its minimum at $k \approx 3 \approx \sqrt{n}$, which slowly shifts in the direction of the smaller k by measure of increase of β. Simultaneously the depth of the minimum somewhat increases. For the considered values of β, the gain, due to the use of the optimum number of exceedings k_{opt} as compared with the case of $k = 1$, considered above, constitutes 1.5-2 db, which corresponds to a gain in range by 8-12%.

In Section 3.8 we also formulated the problem of multistage scanning, with which on the following stage are inspected only those sections, in which on the preceding stages the target was detected. In order to have the possibility to judge the gains, obtained with such a method of scanning, and the variations of parameters of systems of detection connected with this method, let us

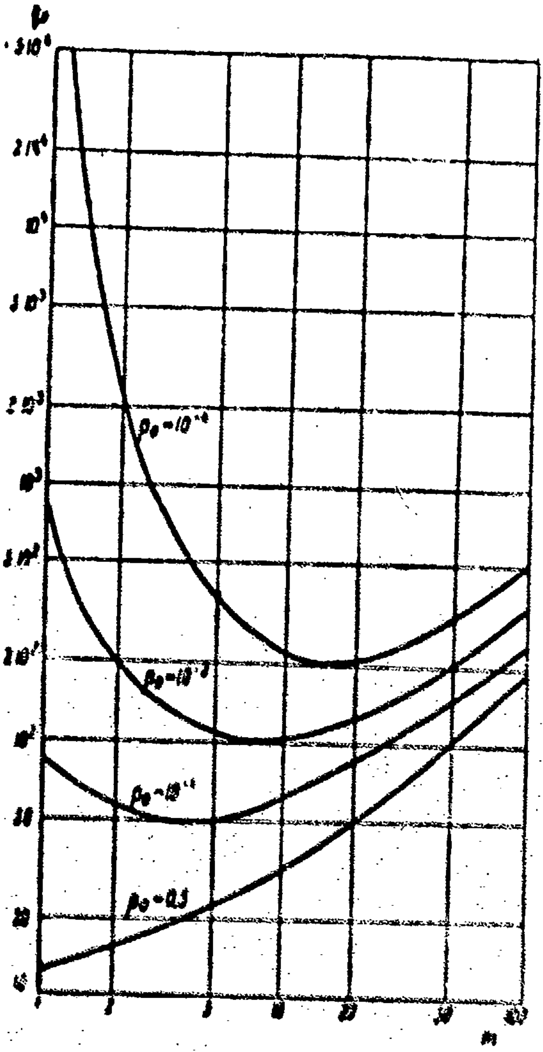

Fig. 4.17. Dependence of threshold signal-to-noise ratio on the number of cycles with an uncoordinated filter band.

consider the case of two-stage scanning with slow fluctuations of reflected signal.

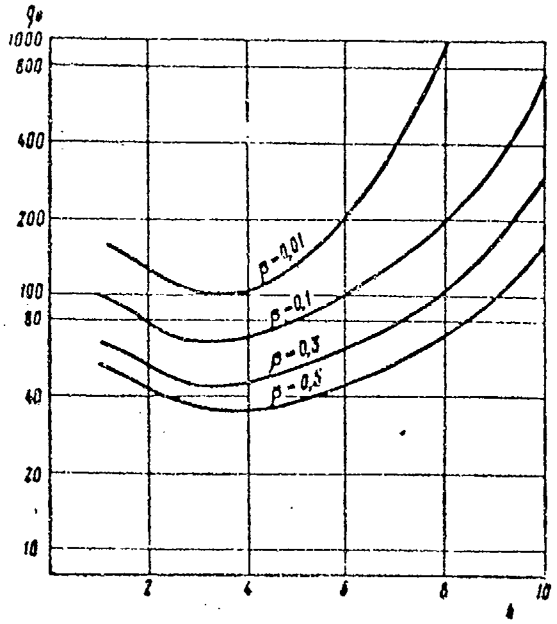

Fig. 4.18. Dependence of threshold signal-to-noise ratio on the required number of operations k in 10 independent cycles.

In accordance with (3.8.4) and (4.4.7), considering $|C_1(\tau_1-\tau)|^2=1$ in (4.4.7), we obtain an expression for probability of miss

$$B_2(F_1, T)=\left(1-F_1^{\frac{1}{1+qT_1}}\right)+F_1^{\frac{1}{1+qT_1}}\left(1-F_2^{\frac{1}{1+qT_2}}\right),$$

which, taking into account (3.8.5), (3.8.6) can be rewritten in the form

$$B_2(F, T)=\left(1-F_1^{\frac{1}{1+qT_1}}\right)+ \\ +F_1^{\frac{1}{1+qT_1}}\left[1-\left(\frac{F}{F_1}\right)^{\frac{1}{1+q\frac{T-T_1}{F_1}}}\right]. \tag{4.7.4}$$

where $q = \frac{q_c}{T} = \frac{P_c}{2N_0}$ is the ratio of power of signal to spectral density of noise.

Considering the probability of miss on the first and second stages to be small (only with such conditions can there be obtained a small general probability of miss) and considering (4.4.22), instead of (4.7.4) it is possible to write

the following approximate equality:

$$B_2(F,T)\approx\frac{1}{q_0}\left[\frac{\ln\frac{1}{F_1}}{\frac{T_1}{T}}+\frac{F_1\ln\frac{F_1}{F}}{1-\frac{T_1}{T}}\right], \tag{4.7.4'}$$

where q_0 as before, is equal to

$$qT=\frac{P_cT}{2N_0}.$$

Equating to zero the derivatives $\frac{\partial B_2}{\partial F_1}$ and $\frac{\partial B_2}{\partial T_1}$, we obtain the following equations for values of F_1 and T_1, with which the probability of miss is minimum:

$$\frac{T}{T_1}-1=\sqrt{\frac{F_1\ln\frac{F_1}{F}}{\ln\frac{1}{F_1}}},\quad F_1\left(1+\ln\frac{F_1}{F}\right)=\frac{T}{T_1}-1. \tag{4.7.5}$$

Excluding $\frac{T}{T_1}$ from these equations, we have

$$\frac{1}{\sqrt{\ln\frac{F_1}{F}}}+\sqrt{\ln\frac{F_1}{F}}=\frac{1}{\sqrt{F_1\ln\frac{1}{F_1}}}. \tag{4.7.6}$$

The magnitude of the root of this equation interesting to us at various F, and also the corresponding values of $\frac{T_1}{T}$ are presented in Table 4.1.

Table 4.1

F	$10^{-[illegible]}$	$10^{-[illegible]}$	$10^{-[illegible]}$	$10^{-[illegible]}$	10^{-10}	$10^{-[illegible]}$	$10^{-[illegible]}$
F_1	$10^{-[illegible]}$	$4.10^{-[illegible]}$	$2.5.10^{-[illegible]}$	$1.6.10^{-[illegible]}$	$1.2.10^{-[illegible]}$	$9.10^{-[illegible]}$	$8.10^{-[illegible]}$
$\frac{T_1}{T}$	0.8	0.8	0.8	0.8	0.8	0.8	0.8
$\frac{q_0'}{q_0}$	0.87	0.57	0.41	0.34	0.3	0.25	0.22

In the table also are presented the values of relative decrease of threshold signal-to-noise ratio, due to the use of two-stage scanning. As can be

seen, two-stage scanning allows to receive an essential gain in the free-space range with the same average time of scanning, whereby the magnitude of gain increases by measure of decrease of probability of false alarm F. It is interesting to note that the ratio $\frac{T_1}{T}$, ensuring minimum of probability of miss, almost does not change with the change of F $\frac{T_1}{T} = 0.76$ at $F = 10^{-2}$; $\frac{T_1}{T} = 0.82$ at $F = 10^{-}$).

Comparison of two-stage scanning with the case of two independent cycles shows that for $\beta \sim 0.1$ and less, two-stage scanning gives an essential loss in distance at the same average time of scanning. Apparently, it is more expedient to use this method of scanning with statistically dependent signals, taken on separate stages. For that, it is sufficient to change to the following stage immediately after termination of the preceding. Multistage scanning can, obviously, be produced cyclically. The calculation of characteristics of detection, taking into account fluctuations, for that case is connected with significant difficulties. For a regular signal with known phase during two-stage scanning, calculation of characteristics of detection was conducted in [11], whereupon in this case the gain, as compared with single-stage scanning, was significantly greater than in the case of a fluctuating signal during statistically independent stages, which indirectly confirms the expressed above assumption. On the whole, this question, undoubtedly, needs study in the future.

In conclusion, let us consider one question, connected with searching and also requiring the use of specific characteristics of detection. In Section 3.9, was the proposed and analyzed method of searching with expanded cycles, where it was shown that in optimum distribution of sections by cycles, this method is close in effectiveness to the optimum method of examination of sections. Examination was conducted at given values of probability of miss of target and averge time of delay in every section. Using the specific characteristics of detection, it is possible to also select these magnitudes in such a way that

the average time of searching is minimum. We solved this problem for the particular case of exponential a priori distribution and slow fluctuations of reflected signal, considering, as everywhere in the chapter, the delay time in the section independent of the accepted realization. Using expression (3.9.16) for average time of searching at $x=x_{\text{опт}}\approx\ln\frac{1}{\beta}$, we obtain

$$\bar{t}\approx\bar{\mu}\,\frac{\ln\frac{1}{F}}{q}\,\frac{1}{\beta}\left(1+\ln\frac{1}{\beta}\,\frac{\beta[2-\beta(1+\beta)]}{(1-\beta)^{2}}\right). \qquad (4.7.7)$$

The probability of false alarm F in formula (4.7.7) also depends on the time of delay in the section, if the average frequency f of false alarm is given. However, this dependence is, due to the smallness of f, weak and it is possible to disregard it.

It then remains to find min $\bar{t}$ by β. The minimum takes place at $\beta = 0.27$. Here,

$$\tau\approx 4\,\frac{\ln\frac{1}{F}}{q},\quad \bar{t}\approx 9{,}2\bar{\mu}\,\frac{\ln\frac{1}{F}}{q}. \qquad (4.7.8)$$

The degree of criticality of the found optimum is confirmed by the dependence presented in Table 4.2 of $\xi=\frac{q\bar{t}}{\bar{\mu}\ln\frac{1}{F}}$ on β in the vicinity of the optimum.

It is interesting to emphasize that the optimum β lies in the region of comparatively large probabilities of miss, at the time when usually in practice they try to make this probability possibly small, due to the fact that, as it is now explained, the average time of searching is considerably increased.

Table 4.2

β	0,05	0.1	0,2	0,3	0,4	0,5
ξ	19	16	10,8	9,5	10,7	11,8

Conducted in this paragraph, the consideration of separate problems, relating to the problem optimization of scanning and searching, showed that in a whole number of cases, rational selection of the method of scanning or searching and parameters of the system, carrying out scanning or searching, allows to receive an essential gain in the

free-space range of the target and in the time, expended for detection. On the whole, this question, as already was repeatedly noted, at present is hardly investigated, which, in a significant degree is explained by the mathematical difficulties, with which its solution is connected. The small results received indicates the perspectivity of further investigations in this area and at the same time, they themselves have a sufficiently large applied value.

4.8. Detection of One Form of Signal with Non-Gaussian Law of Distribution

In connection with expansion of the area of use of radar, lately considerable attention has been attracted by problems of detection and measurement of coordinates of space objects (for example, artificial earth satellites). A signal from such objects fluctuates basically due to the unstabilized rotation of the object during flight. The distributive law of fluctuations can considerably differ from normal. Usually the time of rotation on the width of a lobe of a pattern of secondary radiation can be considered large as compared with the time, utilized for detection of an object in each cycle of scanning. Here, for calculation of characteristics of detection it is sufficient to know the one-dimensional distributive law of a reflecting surface (power of reflected signal).

As was noted in chapter 1, this law can be approximated by linear combination of exponential distributions. Such an approximation corresponds to the assumption that at equiprobable aspects, the distribution of power within the limits of each group of lobes of the pattern of secondary radiation is near to exponential (which corresponds to normal distribution for fluctuations of a signal). The coefficient at the exponent represents the probability that the object will be turned towards the radar namely by this group of lobes.

In connection with this, the considered reflected signal may be treated as a normal random process with random power, taking one of the values of $P_{c1}, P_{c2}, \ldots, P_{cn}$ with probabilities $p_1, p_2, \ldots, p_n$ accordingly, and in the solution of

the problem of detection of such a signal, to use the results, received for a signal, distributed according to normal law.

Inasmuch as the form of optimum operations on the signal during slow fluctuations does not depend on the distributive law of amplitude of signal, an optimum system of detection of the considered signal coincides with that synthesized in paragraph 4.3.3. The characteristics of detection for that case can be found by averaging the characteristics of detection of the normal signal by all possible values of the signal-to-noise ratio. As a result, we obtain

$$D_0(F)=\sum_{k=1}^{n} p_k D(F,\ q_{0k}). \tag{4.8.1}$$

Substituting (4.4.12) in (4.8.1), we have

$$D_0(F)=\sum_{k=1}^{n} p_k \exp\left\{-\frac{K^{-1}_{2\Delta f_\Phi T}(F)-2(\Delta f_\Phi T-1)}{2(1+q_{0k})}\right\}. \tag{4.8.2}$$

In particular, for coordinated filter ($\Delta f_\Phi T=1$)

$$D_0(F)=\sum_{k=1}^{n} p_k F^{\frac{1}{1+q_{0k}}}. \tag{4.8.3}$$

For an illustration, by the formula (4.8.3) in Fig. 4.19 is constructed the dependence of the probability of correct detection D_0 on the smallest signal-to-noise ratio q_{01} for $n = 3$ and $p_1 = 0.5$; $p_2 = 0.2$; $p_3 = 0.3$; $q_{02} = 20q_{01}$; $q_{03} = 300q_{01}$.

In the same figure, for comparisons, curves are constructed for the probabilities of correct detection of normal signal with signal-to-noise ratios q_{01} and

$$\bar{q}_0=\sum_k p_k q_{0k}\approx 100q_{01}.$$

From the comparison of the curves it is clear that a signal of the considered form is detected better than the normally fluctuating signal with a signal-to-

noise ratio, equal to the lowest value of q_{01}, q_{02}, q_{03}, but significantly worse, than a normally fluctuating signal with the same average signal-to-noise ratio.

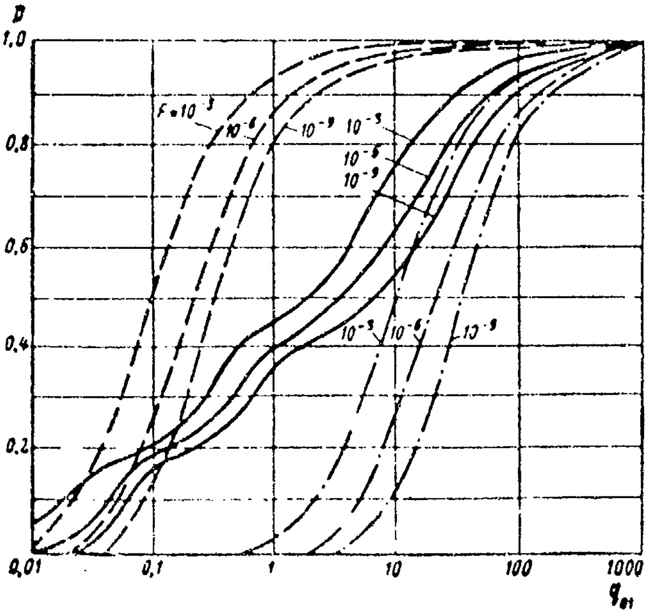

Fig. 4.19. Characteristics of detection of signal with Non-Gaussian distributive law:
—— signal with non-Gaussian distributive law;
- - - normally distributed signal with $q_0 = \bar{q}_0$;
-.-.- normally distributed signal with $q_0 = q_{01}$

The proposed method of calculation of characteristics of detection of signals with non-Gaussian distributive law is, undoubtebly, only a half measure. In order to consider questions of detection of such a signal in intervals, comparable with the time of correlation, and also in order to consider the influence of fluctuations on the accuracy of measurement of coordinates, it is necessary to have a more thorough calculation of properties of these fluctuations.

4.9. Optimum Detection of Radar Signals in the Presence of Passive Interferences

4.9.1. Optimum Processing of Received Signal

This and the several following paragraphs we shall devote to the very urgent contemporary radar problem of detection of a signal from a target on a background of interfering reflections. The cause of such reflections can be a cloud of

metallized dipoles, the surface of the land or sea, individual interfering targets and so forth. Under some limiting assumptions, the influence of reflections can be investigated in general form, not specifying their physical nature. First of all we will assume (see Section 1.3) that an interfering reflected signal, as a signal from a target, is a normal random process with zero mean value. The function of ocrrelations of this signal in accordance with the results in Chapter 1 will be considered in the form

$$R_{nn}(t_1, t_2) = \mathrm{Re}\int_0^\infty \sigma(x)\, u(t_1 - x)\, u^*(t_2 - x) \times$$
$$\times e^{j[\omega_0 + \omega_{\text{д}}(x)](t_1 - t_2)}\, r(x, t_1 - t_2)\, dx, \qquad (4.9.1)$$

where $\sigma(x)$ is the density of intensity of interference, corresponding to the given value of delay x;

$\omega_{\text{д}}(x)$ is the Doppler frequency;

$r(x, t)$ is the coefficient of correlation of interference fluctuations, also dependent, in general, on x.

Presentation (4.9.1) of the function of correlation of interference is correct for a very large number of cases. The conditions of validity of this presentation were discussed in detail in Section 1.3. As in the examination of the signal from the target, we shall assume that fluctuations of interfering reflections, described by the coefficient of correlation r(x, t), are quite slow and do not distort considerably the law of modulation of reflected signal.

Inasmuch as interference is distributed according to the law of Gauss, in the synthesis of an optimum system of detection and investigation of the characteristics of detection it is possible to completely use the general relationships in Section 4.2. We shall begin the consideration from the case of slow fluctuations of a useful signal. Here, the solution of equation (4.2.4) with substitution in it instead of $R_c(t_1, t_2)$ the formula (1.3.9), as we can be convinced of, has the form

$$V(t_1, t_2) = \frac{2P_c}{1+q_1} \operatorname{Re} Z(t_1, \tau) Z^*(t_2, \tau), \tag{4.9.2}$$

where

$$Z(t, \tau) = \int_0^T W_{п}(t, s) u(s-\tau) e^{i\omega_1 s} ds; \tag{4.9.3}$$

$$q_1 = \frac{1}{2} P_c \int_0^T Z^*(s, \tau) u(s-\tau) e^{i\omega_1 s} ds. \tag{4.9.4}$$

As can be seen from (4.9.2), optimum processing of the received signal in this case reduces to multiplication by the reference signal Z(t, τ), occuring as the result of definite integral conversion of the law of modulation of the main signal, and integration of the received product during the time of observation. In order to receive the reference signal in clear form, it is necessary to solve the equation for $W_{п}(t_1, t_2)$ and to substitute the solution in (4.9.3).

The case of fast fluctuations of a useful signal will be considered on the assumption that the time of correlation $\tau_{кс}$ of signal fluctuations significantly exceeds the time of correlation of interference $\tau_{пп}$. This assumption in practice usually is executed and at the same time allows to connect the solution of the problem for the case of fast fluctuations with the solution, found for the case of slow fluctuations of a reflected signal. With this assumption, the solution of equation (4.2.4) naturally, as we did more than once, be found in quasidegenerate form

$$V(t_1, t_2) = \operatorname{Re} Z(t_1, \tau) Z^*(t_2, \tau) \rho(t_1 - t_2), \tag{4.9.5}$$

where Z(t, τ) is determined by equality (4.9.3), and function $\rho(t)$ is assumed just as slow, as the coefficient of correlation of fluctuations of the reflected signal $\rho(t)$.

Let us assume that there exists the limit

$$h_1 = \lim_{T\to\infty} \frac{P_c}{2N_1 T} \int_0^T Z^*(t, \tau) u(t-\tau) e^{i\omega_1 t} dt, \tag{4.9.6}$$

where Δf_c the effective width of the spectrum of signal fluctuations.

Physically, in a sufficient measure, it is obvious that, inasmuch as both the function of correlation of interference, and also $Z(t, \tau)$ must possess, in a known measure, the properties of the law of modulation, this limit exist in all those cases, when there exists a limit $C(\tau, 0)$ in (1.2.2'). After we explain in more detail the properties of $Z(t, \vartheta)$, this will become clearer. If limit (4.9.6) exists and $\tau_{\text{кп}} \ll \tau_{\text{кс}}$, then the equation for $v(t)$, obtained by substitution of (4.9.5) in (4.2.4), has the form

$$v(t_1 - t_2) + h_1 \Delta f_c \int_0^T v(t_1 - t)\rho(t - t_2)\,dt = P_c \rho(t_1 - t_2), \tag{4.9.7}$$

coinciding with (4.3.5). At $\Delta f_c T \gg 1$, equation (4.9.7), as equation (4.3.5), may be solved by Fourier transform. The obtained result, with an accuracy up to immaterial constant factor $\frac{P_c N_0}{\Delta f_c h_1}$ coincides with (4.3.6), if in this formula h is replaced by h_1. Hence, also from formulas (4.9.2) and (4.9.5) it directly follows that optimum processing of the received signal in the case of the presence of passive interferences can be carried out with the help of diagrams of the form, shown in Fig. 4.2-4.4, in which only the form of the reference signal changes. Thus, for a final understanding of the essence of optimum operations we must only definitize the form of this signal.

The solution of equation (1.4.2) at $R(t_1, t_2) = R_n(t_1, t_2) = N_0\delta(t_1, t_2) + R_{na}(t_1, t_2)$ on the strength of the assumption about the slow change of $r(x,t)$ has meaning to find in the form

$$\begin{gathered} W_n(t_1, t_2) = \frac{1}{N_0}\delta(t_1 - t_2) - \\ - \operatorname{Re}\int_0^\infty\int_0^\infty w(x_1, x_2; t_1, t_2)\,u(t_1 - x_1)\,u^*(t_2 - x_2) \times \\ \times e^{i\omega_d(x_1)t_1 - i\omega_d(x_2)t_2}\,dx_1\,dx_2\,e^{i\omega_0(t_1 - t_2)}, \end{gathered} \tag{4.9.8}$$

where $w(x_1, x_2; t_1, t_2)$ is considered to be a slow function of t_1 and t_2 as compared with $u(t)$. By substituting (4.9.8) in the equation and equating coefficients at $u(t_1 - x_1)\, u^*(t_2 - x_2) \times e^{i\omega_d(x_1)t_1 - i\omega_d(x_2)t_2}\, dx_1 dx_2$ in both parts of the

equality, we obtain an equation for $w(x_1, x_2; t_1, t_2)$ in the form

$$w(x_1; x_2; t_1, t_2)+\frac{\sigma(x_2)}{2N_0}\int_0^T dt\int_0^\infty w(x_1, x; t_1, t)\times$$
$$\times e^{-i[\omega_{\text{д}}(x)-\omega_{\text{д}}(x_2)]t}\, r(x_2, t-t_2)\, u^*(t-x)\times$$
$$\times u(t-x_2)\,dx=\frac{1}{N_0^2}\sigma(x_1)\, r(x_1, t_1-t_2)\,\delta(x_1-x_2). \tag{4.9.9}$$

In the case of complex-modulated single sending, assuming $r(x, t_1-t_2) = 1$, we obtain

$$w(x_1, x_2)+\frac{\sigma(x_1)\,T_{\text{эф}}}{2N_0}\int_0^\infty w(x_1, x)\,C[x-x_2, \omega_{\text{д}}(x)-$$
$$-\omega_{\text{д}}(x_2)]\,dx=\frac{1}{N_0^2}\sigma(x_2)\,\delta(x_2-x_1). \tag{4.9.10}$$

In the case of a periodic signal, considering $\omega_{\text{д}}(x)=\omega_{\text{д}}=$ const and averaging the product $u^*(t-x)u(t-x_2)$ by t under the integral, we find

$$w(x_1, x_2; t_1, t_2)+$$
$$+\frac{\sigma(x_2)}{2N_0}\int_0^T dt\int_0^\infty w(x_1, x; t_1, t)\,C(x-x_2)\,r(x_2, t-t_2)\,dx=$$
$$=\frac{1}{N_0^2}\sigma(x_1)\,r(x_1, t_1-t_2)\,\delta(x_1-x_2). \tag{4.9.11}$$

These simplified equations can be solved for the case of extensive interference, when $\sigma(x)$ changes little in the interval of solution by distance, and for the case, when interference is created by a finite number of pointed reflecting objects.

Still not resorting to the solution of these equations, but using only presentation (4.9.8), it is possible to perceive some essential peculiarities of optimum processing in the presence of passive interferences. By substituting (4.9.8) in (4.9.3), we obtain

$$Z(t, \tau)=\frac{1}{N_0}\Big[u(t-\tau)\,e^{i\omega_0 t}-$$
$$-\int_0^\infty f(t, \tau, x)\,u(t-x)\,e^{i[\omega_0+\omega_{\text{д}}(x)]t}\,dx\Big]. \tag{4.9.12}$$

where

$$f(t, \tau, x) = \frac{N_0}{2}\int_0^T dt_1 \int_0^\infty w(x, x_1;\ t,\ t_1)\, u(t_1 - \tau)\, u^*(t_1 - x_1) \times$$
$$\times e^{i[\omega_{\text{Д}0} - \omega_{\text{Д}}(x_1)]t_1} dx_1; \qquad (4.9.13)$$

$\omega_{\text{Д}0} = \omega_1 - \omega_0$, is the Doppler shift of signal from target;

τ is the delay of this signal.

From (4.9.12) and (4.9.13), the meaning optimum operations is obvious. The first component in (4.9.12) ensures optimum separation of useful signal on a background of noises. Multiplication of y(t) by $u(t - x)e^{i[\omega_0 + \omega_{\text{Д}}(x)]t}$ in the second component ensures optimum separation of interference, corresponding to the value of delay x. The separated signals of interference are summarized by all x with weight f(t, τ, x) and are subtracted from the results of multiplication of y(t) by the expected signal from the target. Thus is carried out compensation of interference.

The weight function f(t, τ, x) diminishes as with the growth of $|\tau - x|$ and $|\omega_{\text{Д}0} - \omega_{\text{Д}}(x_1)|$ the signal from the target and interfering reflections become all the more orthogonal. Indeed, as one may see from (4.9.10) and (4.9.11), function w(x_1, x_2; t_1, t_2) diminishes as C(x_1—x_2) tends to zero with the increase of $|x_1 - x_2|$. This follows from the uniqueness of the solution of these integral equations and from the fact that the functions, possessing the shown properties, can satisfy them. Upon subtraction in (4.9.12) the components of the law of modulation of the signal from the target are suppressed, the most characteristic for interference: the reference signal, as it were, is orthogonalized in reference to the interference. This peculiarity of optimum processing is developed most clearly in the case of "discrete" interference (interference of the "interfering target" type).

4.9.2. Case of Discrete Interference

In this case

$$\sigma(x) = \sum_{j=1}^{m} \sigma_j \delta(x - \tau_j).$$

Substituting in (4.9.9)

$$w(x_1, x_2; t_1, t_2) = \sum_{j,k} w_{jk}(t_1, t_2)\, \delta(x_1 - \tau_j)\, \delta(x_2 - \tau_k), \tag{4.9.14}$$

we obtain a system of equations for $w_{jk}(t_1, t_2)$

$$w_{jk}(t_1, t_2) + \frac{\sigma_k}{2N_0} \sum_{l=1}^{m} \int_0^T w_{jl}(t_1, t)\, r_k(t - t_2)\, u^*(t - \tau_l) \times$$
$$\times u(t - \tau_k)\, e^{-i(\omega_{2l} - \omega_{2k})t}\, dt = \frac{\sigma_k}{N_0^2}\, r_k(t_1 - t_2)\, \delta_{jk}, \tag{4.9.15}$$

where

$$r_k(t) = r(\tau_k, t);\; \omega_{2k} = \omega_2(\tau_k).$$

If fluctuations of all interfering targets are slow, then $r_k(t) \equiv 1$ and $W_{jk}(t_1, t_2) = W_{jk}$ does not depend on t_1, t_2. System (4.9.15) is then converted to the form

$$\sum_{l=1}^{m} w_{jl} \left(\delta_{lk} + \frac{\sigma_k T}{2N_0} C_{lk}\right) = \frac{1}{N_0^2}\, \delta_{jk}, \tag{4.9.16}$$

where

$$C_{lk} = C(\tau_l - \tau_k, \omega_{2k} - \omega_{2l})\, e^{i(\omega_{2k} - \omega_{2l})\tau_l}.$$

With a large number of interfering targets, the solution of this system, although not connected with principal difficulties, is rather laborious. The expression for Z(t, τ) may be written in the form

$$Z(t, \tau) = \frac{1}{N_0} \Big[u(t - \tau)\, e^{i\omega t} -$$
$$- \sum_{j,k} v_{jk} C_{k0}\, u(t - \tau_j)\, e^{i(\omega_0 + \omega_{2j})t} \Big]. \tag{4.9.17}$$

where $C_{k0} = C(\tau_k - \tau, \omega_{д0} - \omega_{дk})e^{i(\omega_{д0}-\omega_{дk})\tau_k}$ characterizes the degree of nonorthogonality of signals from detected and k^th interfering targets, and $\|v_{jk}\|$ — the matrix, inverse to $\left\|C_{jk} + \frac{2N_0}{q_k T_{эф}}\delta_{jk}\right\|$. These results are obtained by another method in [64].

The case of fast fluctuations of interfering targets may be considered with a periodic signal and $\omega_{дj}$ = const. Here, producing in (4.9.15) averaging of modulation by period and converting both parts of the equation by Fourier, we obtain for spectra $W_{jk}(\omega)$ of functions $w_{jk}(t_1, t_2)$, a system of equations.

$$\sum_{l=1}^{m} W_{jl}(\omega)\left[\delta_{lk} + \frac{q_k}{2N_0}C'_{lk}S_k(\omega)\right] = \frac{q_k}{N_0^2}S_k(\omega)\delta_{jk}, \tag{4.9.18}$$

where $C'_{lk} = C_0(\tau_l - \tau_k, 0)$ (see Section 1.2);

$S_k(\omega)$ is the spectral density of fluctuations from k^th interferring target.

Equation (4.9.18) for spectra $W_{jk}(\omega)$ in form is fully analogous to equation (4.9.16). Considering that $w_{jk}(t_1, t_2)$ during fast fluctuations depends only on the difference $t_1 - t_2$ and that according to the assumptions, this function changes little in the period of repetition $T_п$, we have, by replacing the limits of integration in (4.9.13) with infinite ones:

$$\begin{aligned} I(t,\tau,x) &= \frac{N_0}{2}\sum_{j,k}\delta(x-\tau_j)C'_{kj}\ldots \\ &\times e^{i\Delta\omega_k t}\sum_{l=-\infty}^{\infty} w_{jk}(lT_п)e^{-il\Delta\omega_k T_п} \ldots \\ &= \frac{N_0}{2}\sum_{j,k}\delta(x-\tau_j)C'_{kj}\ldots \\ &\times e^{i\Delta\omega_k t}\sum_{l=-\infty}^{\infty} W_{jk}(\Delta\omega_k + l\Omega_п), \end{aligned}$$

where $\Delta\omega_k$ is the difference of Doppler frequencies of the target and interference;

$$\Omega_п = \frac{2\pi}{T_п}.$$

In the derivation of this formula, the integral by t_1 in (4.9.13) was

represented in the form of the sum of integrals by periods and $w_{jk}(t)$ in each period was considered constant. Furthermore, we used the connection between the spectrum of sequence $w_{jk}(lT_r)$ and spectrum of function $w_{jk}(t)$ (see [67]).

Substituting $f(t, \tau, x)$ in (4.9.12), we obtain

$$Z(t, \tau) = \frac{1}{N_0}\left[u(t-\tau) - \sum_{j,k} v'_{jk} C'_{k0} u(t-\tau_j)\right] e^{i\omega_0 t}, \tag{4.9.19}$$

where

$$v'_{jk} = \sum_{l=-\infty}^{\infty} W'_{jk}(\Delta\omega_{\text{д}} + l\Omega_r),$$

$\|W'_{jk}(\omega)\|$ is the matrix, inverse to

$$\left\| C'_{jk} + \frac{2N_0}{\sigma_k S_k(\omega)} \delta_{jk} \right\|.$$

In order to make a more graphic meaning of optimum conversions of received signal, determined by formulas (4.9.18), (4.9.19), it is convenient to somewhat convert these formulas.

We shall add to matrix $\left\| C_{jk} + \frac{2N_0}{\sigma_k T_{\text{эф}}} \delta_{jk} \right\|$ a zero column $C_{0k} + \frac{2N_0}{\sigma_k T_{\text{эф}}} \delta_{0k}$ and line $C_{j0} + \frac{2N_0}{\sigma_0 T_{\text{эф}}} \delta_{j0}$. Here, the order of the matrix becomes equal to m+1. Using the known presentation of inverse matrix by cofactors of its elements and considering that the considered matrices are Hermitian, it is possible to show that the elements of inverse matrices m and $(m+1)^{th}$ order are connected by the relationship

$$v_{j0}^{(m+1)} = -v_{00}^{(m+1)} \sum_{k=0}^{m} v_{jk}^{(m)} C_{k0}. \tag{4.9.20}$$

Using (4.9.20), reference signal $Z(t, \tau)$, determined by formula (4.9.18), may be written in the form

$$Z(t, \tau) = \frac{1}{N_0 v_{00}^{(m+1)}} \sum_{j=0}^{m} v_{j0}^{(m+1)} u(t-\tau_j) e^{i\omega_{\text{д}j} t + i\omega_0 t}. \tag{4.9.21}$$

By τ_0 and $\omega_{\text{д}0}$ in this formula, we mean delay and Doppler frequency of signal from target.

In a similar manner will be converted formula (4.9.19). Thus, in the case of slow and fast fluctuations, the optimum reference signal is a linear combination of expected signals from all targets. If the signal-to-noise ratio is great for all targets $\left(\frac{\sigma_k T_{ob}}{2N_0} \gg 1\right.$ at all k), then matrix $\|v_{jk}^{(m+1)}\|$ can be considered equal to the matrix, inverse to $\|C_{jk}\|_0^m$. Here, multiplication by the reference signal and integration obtain the most simple meaning: this operation ensures full suppression of signals from interfering targets. Indeed, the reference signal in this case is completely orthogonal to signals from interfering targets:

$$\int_0^T Z(t,\tau)\,u^*(t-\tau_k)\,e^{-i\omega_0 t - i\omega_{\partial k} t}\,dt = \sum_{j=0}^{m} v_{j0}^{(m)} C_{kj} = 0.$$

In those cases, when we cannot disregard noise, processing of the considered form ensures incomplete suppression of interfering signals, decreasing them approximately to the level of noise (see Section 4.10). Simultaneously is suppressed the useful signal and the signal-to-noise ratio decreases.

4.9.3. Case of Extensive Interference

Let us now turn to the case of extensive interference, when the density of interference and its properties hardly change in the interval of delays, corresponding to the interval of solution by distance. In the synthesis of an optimum system of processing of received signal for that case it is possible to use expressions for function of correlation (1.3.12) and (1.3.13), corresponding to the case of stationary interference. This follows from the fact that optimum processing in the general case [see (4.9.12)] includes multiplication by the expected signal from the target and the totality of expected signals from interference and integration, whereby the weight function $f(t,\tau,x)$ diminishes by measure of increase of $|\tau-x|$ to zero, approximately as $C(x-\tau)$. Thus, reflectors, forming interference and distant from the target in the interval of

delays, significantly greater than the interval of solution, do not affect the character of optimum processing. If the properties of interference slightly change in the interval of solution, then, allowing, for convenience of solution, the smallness of these changes in the interval, ensuring stability of interference, we almost will not change the result. This method, receiving name "stationary continuation of interference", was proposed in [54] without a sufficiently clear logical foundation, which is possible only after clearing up the general properties of optimum operations. The same results are obtained from approximate solution of equations (4.9.10) and (4.9.11); however, this way requires rather laborious calculations with that same, in general, degree of conclusiveness.

In accordance with the above, in stationary continuation of interference we should consider $\sigma(x)\, r\,(x,\, t_1 - t_2)$ to be equal to the value of this function at $x = \tau$. In the case of single sending, from (1.3.12) we obtain

$$R_{пп}(t_1,\, t_2) \approx \operatorname{Re} \sigma(\tau)\, r\,(\tau,\, t_1 - t_2)\, T_{эф} C\,[\alpha\,(t_1 - t_2)] \times \\ \times e^{i(\omega_0 + \omega_{дп})(t_1 - t_2)}, \tag{4.9.22}$$

where $\omega_{дп}$ is the Doppler shift of frequency of interference, which we will consider not depending on delay (for extensive interference this is usually completed).

Analogously, in the case of a periodic signal, from (1.3.13) we have

$$R_{пп}(t_1,\, t_2) \approx \operatorname{Re} \sigma_0(\tau)\, T_r r_0(\tau,\, t_1 - t_2) \times \\ \times C_0[\alpha\,(t_1 - t_2)]\, e^{i(\omega_0 + \omega_{дп})(t_1 - t_2)}, \tag{4.9.23}$$

where

$$\sigma_0(\tau)\, r_0(\tau,\, t_1 - t_2) = \sum_{l=0}^{\infty} \sigma\,(\tau + lT_r)\, r\,(\tau + lT_r,\, t_1 - t_2).$$

Summation in the last expression considers the circumstance that with a periodic signal, on the strength of the ambiguity inherent to this signal, there occurs imposition of signals from objects, distant from each other at a magnitude,

$U_0(\omega)$ — is the spectrum of one period of modulation.

Coefficient α in these formulas is considered the Doppler effect at the frequency of repetition.

It is necessary to note that formulae (4.9.25) and (4.9.26) are not mutually exclusive. A single sending, with which we were concerned in the derivation of formula (4.9.25), can, in particular, represent a group of periods of modulation. However, so that the conditions of derivation of this formula are satisfied, it is necessary to demand, considering the imposition of signals with value of delay, multiple T_r, so that $\sigma(\tau) \approx \sigma(\tau + lT_r)$ at $-\frac{T_{эф}}{2T_r} < l < \frac{T_{эф}}{2T_r}$ and $\sigma(\tau)$ hardly changes in the vicinity of these points. Thus, the distinction of the considered formulas consists in the fact that formula (4.9.26) is taken specially for a periodic signal under limitations, less rigid, than those, which are required in the application of formula (4.9.25) to this case.

Let us now turn to the investigation of properties of the reference signal for the considered case. Substituting (4.9.25) in (4.9.24) and (4.9.3), we obtain

$$Z(t, \tau) = \frac{e^{i\omega_0 t}}{2\pi N_0} \int_{-\infty}^{\infty} \frac{U_T(\omega)\, e^{i\omega(t-\tau)}}{1 + \frac{1}{2N_0} S(\omega + \Delta\omega_\text{д})}\, d\omega, \tag{4.9.27}$$

where $U_T(\omega)$ is the spectrum of expected signal, calculated in interval (0,T),

$\Delta\omega_\text{д} = \omega_{\text{д}0} - \omega_{\text{дп}}$ is the difference of Doppler shifts of useful signal and interference.

As can be seen from this formula, the reference signal is obtained by transmission of the expected signal from the target through a rejector filter, suppressing, in this signal, the spectral component, most clearly represented in the spectrum of interference. The frequency response of this filter in the majority of cases is physically not realizable. Therefore, it is necessary to find reasonable physically realizable approximations for this response.

Let us examine the case of complex-modulated single sending, for which

multiple $\frac{cT_r}{2}$ (T_r —period of repetition).

In the case of signals, ensuring high resolving power in distance with the usual correlation processing (Section 4.3), the limitations, placed on interference during stationary continuation, usually are carried out. For systems with continuous emission without modulation the conditions of stability, as one may see from (1.3.7), are always fulfilled.

Solution of equation (1.4.2) during stationary interference and time of observation T, large as compared with time of correlation of interference, can be received by Fourier transform

$$W_{\text{п}}(t_1, t_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t_1 - t_2)}}{N_0 + S_{\text{пп}}(\omega)}\, d\omega, \qquad (4.9.24)$$

where $S_{\text{пп}}(\omega)$ is spectral density of passive interference.

In connection with the presence of high-frequency factor in (4.9.22) and (4.9.23) $S_{\text{пп}}(\omega)$ is represented in the form

$$\frac{1}{2}\left[S(\omega - \omega_0 - \omega_{\text{дп}}) + S^*(-\omega - \omega_0 - \omega_{\text{дп}})\right].$$

In the case of single sending, from (4.9.22) we find

$$S(\omega) = \sigma(\tau)\, T_{\text{эф}}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\text{м}}\left(\frac{\omega - \omega'}{a}\right) S_{\text{ф}}(\tau, \omega')\, d\omega', \qquad (4.9.25)$$

where $S_m(\omega)$ is the spectrum of function $C(\tau)$, , which in the future we shall call spectral the density of modulation;

$S_{\text{ф}}(\tau, \omega)$ is the spectral density, corresponding to the coefficient of correlation $r(\tau, , t_1 - t_2)$.

In the case of a periodic signal

$$S(\omega) = \sigma_0(\tau) \sum_{-\infty}^{\infty} S_{\text{м}0}(k\Omega_r)\, S_{\text{ф}}(\omega - ka\Omega_r) \qquad (4.9.26)$$

where $S_{\text{м}0}(\omega) = \frac{|U_0(\omega)|^2}{T_r}$ is the spectrum of one period of function $C_0(\tau)$;

the function $C(\tau)$ has one sufficiently short maximum. The spectrum $S_{\text{ж}}(\omega)$ in this case appears to be very broad, since its shift by $\Delta\omega_{\text{д}}$ may be disregarded and one rejector filter used for obtainment of reference signals corresponding to various values of Doppler frequencies. Since the duration of the signal is finite, multiplication by the reference signal can be replaced by filtration (see Section 4.3).

In the case of a periodic signal, the frequency response of the filter has the form

$$H_{\text{р}}(i\omega)=\frac{1}{1+\frac{\sigma_0(\tau)}{2N_0}\sum_{-\infty}^{\infty}S_{\text{ж}0}(k\Omega_r)\,S_{\varphi 0}(\tau,\ \omega-k\Omega_r)}. \qquad (4.9.28)$$

Usually, the width of the spectrum of modulation is great in comparison with the frequency of repetition and with the width of the spectrum of fluctuations $S_{\varphi 0}(\tau,\omega)$. Here,

$$H_{\text{р}}(i\omega)\approx\frac{1}{1+\frac{\sigma_0(\tau)}{2N_0}S_{\text{ж}0}(\omega)\sum_{-\infty}^{\infty}S_{\varphi 0}(\tau,\ \omega-k\Omega_r)}. \qquad (4.9.29)$$

The approximate form of the frequency response is shown in Fig. 4.20. If the duration of the expected signal essentially exceeds the correlation time of interference, the spectrum of such a signal, which is lined, has spectral lines significantly more narrow than $S_{\varphi 0}(\tau,\omega)$. Here, an essential role is played only by the values of $H_{\text{р}}(i\omega)$ in points $\Delta\omega_{\text{д}}+k\Omega_r\ (k=-\infty,\dots,\infty)$ and this filter can be replaced by any other, for which the values of the frequency response in the indicated points coincide with the values of $H_{\text{р}}(i\omega)$. In Fig. 4.20 are presented possible approximations of $H_{\text{р}}(i\omega)$ for several values of $\Delta\omega_{\text{д}}$. From the figure it is clear that the dissection of the frequency response of the filter becomes immaterial and the filter may be replaced by a very wide-band one. Here, the time of establishment of voltage in the filter becomes small as compared with the period, the reference signal becomes periodic and processing of the signal is divided into intra-period processing and subsequent storage.

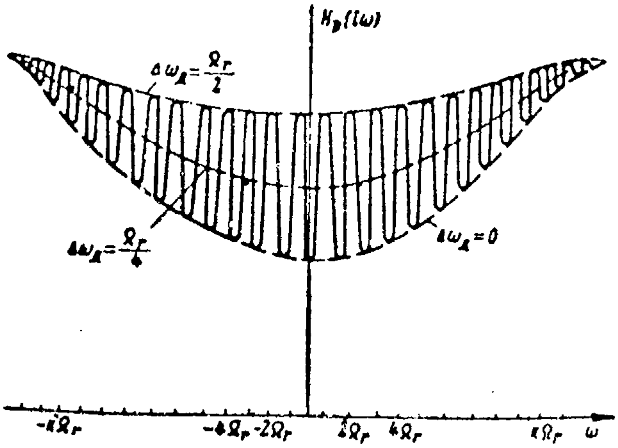

Fig. 4.20. Frequency response of rejector filter.

In the general case, due to the presence of noises, the reference signal $Z(\tau, \omega)$ cannot be represented in the form of the product of a slowly changing function into a periodic function and optimum processing is not divided into intra-period, carried out as if the given period was unique, and inter-period. This circumstance considerably hampers practical realization of optimum processing.

In connection with this, an essential interest is presented by the question of synthesis of optimum inter-period processing in the case, when separate processing of periods is assumed beforehand, and of a comparison of such kind of separate processing with completely optimum (see Section 4.11).

4.10. Characteristics of Target Detection on a Background of Passive Interferences

4.10.1 Equation of Detection Characteristics

As in the case of detection on a background of noise, during the investigation of target detection characteristics on a background of passive interferences, let us consider along with optimum systems a certain quasioptimum, differing from the optimum in its comparative technical simplicity. Calculation of the characteristics of interest to us, as in the case of noise, will be conducted on the basis of the general results of Section 4.2. With this, it will be necessary to consider

separately the case of single transmission of signal -- no assumptions are being made relative to the law of modulation of this transmission -- and the case of a periodic main signal with arbitrary intraperiodic modulation.

When examining the first case, we will assume that fluctuations of the signal are slow, and the filter band ensuring the coherent accumulation of signal conforms to the length of transmission. The support signal utilized in the receiver we will consider arbitrary and designate by $Z_1(t,\tau)\,e^{i\omega_0 t}$. With this the voltage on input of relay is in the form

$$\left|\int_0^T Z_1(t,\tau)e^{i\omega_0 t}y(t)\,dt\right|^2$$

and, inasmuch as y(t) is a normal random process, it is subject to exponential distributive law. The equation of detection characteristics corresponding to this distribution has the form

$$D=F^{\frac{1}{1+q_1}},$$

where q_1 represents the ratio of signal power to total power of noise and interference at output of a system carrying out multiplication by the support signal and integration. Using (4.9.1) and expressing $r(x, t_1-t_2)$ by $S_\phi(x,\omega)$, it is easy to show that

$$q_1=\frac{P_c T_{\text{эф}}}{2N_0}\,\frac{|C_1(x,\tau)|^2}{1+\frac{T_{\text{эф}}}{4\pi N_0}\int\limits_{-\infty}^{\infty}d\omega\int\limits_0^{\infty}|C_1(x,\tau,\Omega_1-\omega)|^2\gamma(x)S_\phi(x,\omega)\,dx},\qquad(4.10.1)$$

where

$$C_1(x,\tau,\omega)=\frac{1}{T_{\text{эф}}}\int\limits_{-\infty}^{\infty}Z_1(t,\tau)u^*(t-x)e^{i\omega t}\,dt,$$

$C_1(x,\tau)=C_1(x,\tau,0)$, and $Z_1(t,\tau)$ is considered for simplicity to be standardized just as u(t).

It is not difficult to see that if the system of detection is optimum and $Z_1(t,\tau)\,e^{i\omega_0 t}=Z(t,\tau)$, determined by formula (4.9.3), then (4.10.1) agrees with (4.9.4).

In the case of the periodic laws of modulation of the main and support signals, as for detection on a background of noises, we consider the dependence of detection characteristics on the filter transmission band and on the width of the fluctuation spectrum of the reflected signal. With certain additional conditions, these dependences, as we now believe, carry the same character as in the cases of detection on a background of noises.

With an arbitrary support signal and filter characteristic, the output voltage of the detection system, as was shown in Section 4.4, can be presented in the form of (4.4.1'). Characteristics of detection are determined by relationships (4.2.6') -- (4.2.8'), where in these relationships the correlation function $r(t_1, t_2)$ in the considered case is equal to

$$r(t_1, t_2)=\frac{P_c}{2N_0}Z_1(t_1,\tau)u^*(t_1-\tau)Z_1^*(t_2,\tau)u(t_2-\tau)\times \tag{4.10.2}$$
$$\times\rho(t_1-t_2)+|Z_1(t,\tau)|^2\delta(t_1-t_2)+$$
$$+\frac{1}{2N_0}R_1^*(t_1,t_2)e^{i\Delta\omega_{\text{д}}(t_1-t_2)}Z_1(t_1,\tau)Z_1^*(t_2,\tau),$$

where $r_1(t_1, t_2)$ designated the coefficient when $\exp[i(\omega_0+\omega_{\text{дп}})(t_1-t_2)]$ in (4.9.1), and $Z_1(t,\tau)$ designates the law of modulation of the support signal which we consider periodic and standarized just as $u(t)$.

Using the assumed slowness of interference fluctuation and the narrow-band nature of the filter, we average $r(t_1, t_2)$ with respect to t_1 and t_2 in an interval sufficient for averaging the modulation of the support and main signals. As a result, using (4.9.1), we obtain

$$\bar{r}(t_1, t_2)\approx\frac{P_c}{2N_0}|C_{10}(\tau,\tau)|^2\rho(t_1-t_2)+\delta(t_1-t_2)+$$
$$+\frac{1}{2N_0}\int\limits_0^\infty\sigma(x)\,dx\,\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}S_\psi(x,\omega-\Delta\omega_{\text{д}})\times$$
$$\times\frac{1}{T}\int\limits_0^T Z_1(t_1+x_1,\tau)u^*(t_1+x_1-x)e^{i\omega(t_1+x_1)}dx_1\times$$
$$\times\frac{1}{T}\int\limits_0^T Z_1^*(t_2+x_2,\tau)u(t_2+x_2-x)e^{-i\omega(t_2+x_2)}dx_2\,d\omega.$$

where

$$C_{10}(x, \tau) = C_{10}(x, \tau, 0);\ C_{10}(x, \tau, \omega) = \frac{1}{T_r}\int\limits_0^{T_r} Z_1(t, \tau)\, u^*(t - x)\, e^{i\omega t}\, dt.$$

Considering $T \gg T_r$, we will break down integrals with respect to t_1 and t_2 into integrals according to periods and we will disregard the boundary effects. As a result of simple conversions we have

$$\widetilde{r(t_1, t_2)} = \frac{P_c}{2N_0}\,|C_{10}(\tau, \tau)|^2 \rho(t_1 - t_2) + \delta(t_1 - t_2) + \frac{1}{4\pi N_0}\int\limits_0^{\infty} \sigma(x)\, dx \int\limits_{-\infty}^{\infty} S_\psi(x, \omega - \Delta\omega_{\text{Д}}) \times e^{i\omega(k - l)T_r} \frac{\sin^2 \frac{\omega T}{2}}{\frac{T^2}{T_r^2}\sin^2\frac{\omega T_r}{2}}\, |C_{10}(x, \tau, \omega)|^2\, d\omega, \tag{4.10.3}$$

where k and l are integral parts of ratios $\frac{t_1}{T_r}$ and $\frac{t_2}{T_r}$, accordingly. Function $\frac{\sin^2\frac{\omega T}{2}}{\frac{T^2}{T_r^2}\sin^2\frac{\omega T_r}{2}}$ has maximums equal to one when $\omega = l\Omega_r$ (l is an integer, $\Omega_r = \frac{2\pi}{T_r}$), the width of which, by assumption, is significantly more than the width of spectrum $S_\psi(x, \omega)$, but significantly less than the frequency of repetition. Considering that $e^{i\omega(k-l)T_r}$ does not change during addition to ω of an integer of repetition frequencies, it is possible to replace the integral with respect to ω in (4.10.3) by

$$\sum_{k=-\infty}^{\infty}\int\limits_{-\infty}^{\infty} S_\psi(x, \omega - \Delta\omega_{\text{Д}} - k\Omega_r)\,|C_{10}(x, \tau, \omega - k\Omega_r)|^2 \times e^{i\omega(t_1 - t_2)}\, d\omega.$$

Due to the slowness of interference fluctuation in this sum, an essential role is played by only the two members with values of k nearest the magnitude of ratio $\frac{\Delta\omega_{\text{Д}}}{\Omega_r}$. Considering the slowness of change $C_{10}(x, \tau, \omega)$ as function ω, it is possible to replace ω in the argument of this function by $\Delta\omega_{\text{Д}}$.

As a result, the final expression for $\widetilde{r(t_1, t_2)}$ is written in the form

$$\widetilde{r(t_1, t_2)} \approx \frac{P_c}{2N_0}\,|C_{10}(\tau, \tau)|^2 \rho(t_1 - t_2) + \delta(t_1 - t_2) + \frac{1}{4\pi N_0}\int\limits_0^{\infty} \sigma(x)\,|C_{10}(x, \tau, \Delta\omega_{\text{Д}})|^2 \times dx \int\limits_{-\infty}^{\infty}\sum_{-\infty}^{\infty} S_\psi(x, \omega - \Delta\omega_{\text{Д}} - k\Omega_r)\, e^{i\omega(t_1 - t_2)}\, d\omega \tag{4.10.4}$$

The process to which the function of correlation $\widetilde{r}(t_1, t_2)$, corresponds is, as one should have expected, stationary and can be characterized by spectral density. With this, we conveniently introduce into consideration the equivalent spectral density of the interference

$$S_{\text{экв}}(\omega) = \frac{1}{2}\int_0^\infty \sigma(x)\,|C_{10}(x, \tau, \Delta\omega_д)|^2 \times$$
$$\times \sum_{-\infty}^{\infty} S_ф(x, \omega - \Delta\omega_д - k\Omega_r)\,dx. \tag{4.10.5}$$

The magnitude of the equivalent spectral density of the interference in the input of the filter considerably depends on the Doppler shift of the interference relative to the useful signal and on the degree of correlation between its law of modulation and the support signal.

The next step in finding the characteristics of detection is the solution of equation (4.2.7') with nucleus

$$v_1(t_1, t_2) = \frac{P_c}{2N_0}|C_{10}(\tau, \tau)|^2\int_0^T v(t_1, t)\rho(t - t_2)\,dt +$$
$$+ v(t_1, t_2) + \frac{1}{N_0}\int_0^T v(t_1, t)\,R_{\text{экв}}(t, t_2)\,dt, \tag{4.10.6}$$

where v(t_1, t) is determined by formula (4.4.2).

During fast fluctuation of signal from target and interference, the solution of this equation can be obtained by Fourier transform. As a result, for the characteristic voltage function at input of the relay, we have

$$\Psi(\eta) = \exp\left\{-\frac{T}{2\pi}\int_{-\infty}^{\infty}\ln\left[1 - i\eta|H(i\omega)|^2\left(1 + \frac{S_{\text{экв}}(\omega)}{N_0} +\right.\right.\right.$$
$$\left.\left.\left. + \frac{P_c|C_{10}(\tau, \tau)|^2}{2\Delta f_c N_0} S_0(\omega)\right)\right]\right\}d\omega. \tag{4.10.7}$$

This formula is fully analogous to formula (4.4.7). Semi-invariants, corresponding to the obtained characteristic function, are determined by expression

$$\varkappa_\nu = (\nu - 1)!\,\frac{T}{2\pi}\int_{-\infty}^{\infty}|H(i\omega)|^{2\nu}\left(1 + \frac{S_{\text{экв}}(\omega)}{N_0} +\right.$$
$$\left. + \frac{P_c|C_{10}(\tau, \tau)|^2}{2N_0\Delta f_c} S_0(\omega)\right)^\nu d\omega. \tag{4.10.8}$$

With sufficiently large $\Delta f_n T$ and $\Delta f_c T$ distribution p(L) can be considered normal [see (4.4.9) and (4.4.10)], or with more accurate calculations we can use Edgeworth series (4.4.11).

If the spectrum of the interference fluctuation Δf_n is wide as compared with the bandwidth of the transmission of the filter Δf_ϕ, the spectral density of interference in the filter band can be considered constant and equal to $S_{nn}(0)$. This corresponds to the replacement of interference by equivalent white noise. Condition $\Delta f_n \gg \Delta f_\phi$ quite often is fulfilled for radar sets, fixed on fast-moving objects, when due to the movement, there occurs a strong expansion of the fluctuation spectrum. Having produced in (4.10.7) a change of variables $\eta' = \left(1 + \frac{S_{nn}(0)}{N_0}\right)\eta$, not reflected in the form of detection characteristics, we obtain

$$\Psi(\eta') = \exp\left\{-\frac{T}{2\pi}\int_{-\infty}^{\infty} \ln\left[1 - i\eta' |H(i\omega)|^2 [1 + h_1 S_1(\omega)]\right] d\omega\right\}, \qquad (4.10.9)$$

where

$$h_1 = \frac{P_c |C_{11}(\tau, \Omega)|^2}{2N_0 \Delta f_c} \cdot \frac{1}{1 + \frac{S_{nn}(0)}{N_0}} = h \frac{|C_{11}(\tau, \Omega)|^2}{1 + \frac{S_{nn}(0)}{N_0}}. \qquad (4.10.10)$$

This formula completely agrees in form with (4.4.7) and from it, it follows that the dependence of the threshold ratio of signal to interference on the filter transmission band, the width of the fluctuation spectrum, and the observation time in the presence of passive interference and when $\Delta f_\phi \gg \Delta f_c$ carries the same character during fast fluctuation of useful signal as in the presence of one noise (see Section 4.4).

We arrive at the same conclusion when considering a case of slow fluctuation of a signal reflected from a target when $\Delta f_c T \ll 1$. For the characteristic function $\Psi(\eta)$ in this case we have

$$\Psi(\eta) = \frac{1 - i\eta\left(1 + \frac{S_{nn}(0)}{N_0}\right)}{1 - i\eta\left[1 + \frac{S_{nn}(0)}{N_0} + \frac{P_c T}{2N_0}|C_{11}(\tau, \Omega)|^2\right]} \times \qquad (4.10.11)$$

$$\times \exp\left\{-\frac{T}{2\pi}\int_{-\infty}^{\infty} \ln\left[1 - i\eta\left(1 + \frac{S_{nn}(0)}{N_0}\right)|H(i\omega)|^2\right] d\omega\right\}.$$

Considering distribution, without a useful signal, normal when $1-D \ll 1$ and $F \ll 1$ we obtain similar to (4.4.19)

$$D \approx \exp\left\{-\frac{\Phi^{-1}(1-F)\sqrt{\varkappa_{2\text{п}}}}{1+\frac{S_{\text{п}0}(0)}{N_0}+\frac{P_c T\,|C_{10}(\tau,\Omega)|^2}{2N_0}}\right\}. \qquad (4.10.12)$$

where $\varkappa_{2\text{п}}$ is determined by the formula (4.10.8) when $P_c = 0$.

If $\Delta f_{\Phi} \ll \Delta f_{\text{п}}$ then by replacing interference by equivalent white noise, we can obtain for this case all the results received for slow fluctuation in paragraphs 4.4.2 and 4.4.4. With this the signal-to-noise ratio q_0 is replaced in the formulas by the signal-to-interference ratio

$$q_1 = q_0\,\frac{|C_{10}(\tau,\Omega)|^2}{1+\frac{S_{\text{п}0}(0)}{N_0}}. \qquad (4.10.13)$$

From the received ratios it is clear that in the presence of passive interference, wide-band as compared with frequency response of the filter, carrying out coherent accumulation in the system of detection, the dependence of threshold power of the signal on the filter band and the relationship between observation time and the width of the fluctuation spectrum of the useful signal have the same form as in the problem of detection on a background of noise.

The dependence of the threshold power of the useful signal on the laws of modulation of the main and support signals is determined by formulas (4.10.10) and (4.10.13). These laws should be selected so that the signal-to-interference ratio q_1 (or proportional to q_1, the signal-to-interference ratio $h_1 = \lim\limits_{T\to\infty}\frac{q_1}{\Delta f T}$) be the largest possible.

When $\Delta f_{\text{п}}$ is comparable with the bandwidth of the filter, the results obtained when $\Delta f_{\Phi} \ll \Delta f_{\text{п}}$ can be used as approximate or definitized with the help of formulas (4.10.9) and (4.10.12) if condition $\Delta f_{\text{п}} T \gg 1$ is fulfilled. To analyze the case $\Delta f_{\text{п}} T \sim 1$ as yet has not been managed because of essential calculating difficulties which are possible to surmount, apparently only with the use of electronic calculating technology. If, however, interference fluctuates slowly ($\Delta f_{\text{п}} T \ll 1$), then solution (4.2.7') can be obtained under the condition of little noise as compared

with interference, which usually is fulfilled. With this,

$$D \approx F^{\frac{1}{1+q_1}},$$

where in the case of a periodic main signal

$$q_1 \approx q_0 \frac{|C_{10}(\tau, v)|^2}{1+\frac{1}{2N_0}\int\limits_0^\infty \sigma(x)|C_{10}(x, \tau, \Delta\omega_d)|^2 dx \sum\limits_{k=-\infty}^{\infty} |H(i\Delta\omega_d - ik\Omega_r)|^2}. \quad (4.10.14)$$

and in the case of a single transmission with arbitrary modulation

$$q_1 \approx q_0 \frac{|C_1(\tau, v)|^2}{1+\frac{1}{2N_0}\int\limits_0^\infty \sigma(x)|C_1(x, \tau, \Delta\omega_d)|^2 dx}. \quad (4.10.15)$$

Here we again arrive at a certain signal-to-interference ratio which is necessary to increase by means of the rational selection of modulation of the support and main signals. Dependence of the signal-to-interference ratio on these laws of modulation is examined below.

4.10.2. Dependence of the Quality of Detection on the Law of Modulation of the Support Signal

The law of modulation of the support signal, corresponding to optimum processing, represents, as was shown in the preceding paragraph, the result of very complicated transformations of the main signal; the character of these transformations depends on the Doppler shift of signal frequency relative to interference, distribution of interference by distance, etc. During a periodic signal, the law of modulation of optimum support signal $Z(t, \tau)$ does not possess, in general, the property of periodicity and optimum processing is not divided into intra- and interperiod. All these circumstances hamper the technical realization of optimum processing and serve as a cause of the tendency to replace the processing by simpler treatment. From this point of view it is most desirable as a support signal to use the expected signal from the target. Such a support signal is optimum during detection on a background of white noise and this replacement would permit us not to be concerned about the change of operating conditions of the radar station in the presence of passive

interferences. Therefore, it is assumed expedient to compare magnitudes of the signal-to-interference ratio during optimum support signal and when using, as the law of modulation of the support signal, the expected law of modulation of the useful signal. For the first of these cases, q_1 can be calculated by the formulas (4.9.4) and (4.10.1) with the use of the results in Section 4.9, concerning determination of optimum support signal. Expression for q_1 during application of the expected signal as support, can be obtained from (4.10.1), (4.10.13), (4.10.14), and (4.10.15), replacing $C_{10}(x, \tau, \omega)$ and $C_1(x, \tau, \omega)$ by $C_0(x-\tau, \omega)$ and $C(x-\tau, \omega)$ respectively.

We conduct comparison of magnitudes q_1 separately for "discrete" and extensive interference.

In the case of discrete interference, substituting (4.9.21) into (4.9.4), we obtain for a single transmission

$$q_1 = \frac{P_c T_{эф}}{2N_0 v_{00}^{(m+1)}} - 1. \qquad (4.10.16)$$

The formula for q_1 during a periodic signal has the same form. The difference consists of the fact that $v_{ээ}^{(m+1)}$ is replaced by $v_{ээ}^{(m+1)}$ (see paragraph 4.9.2).

In order to avoid excessive complications, we will compare optimum processing with the usual correlated processing for a particular case of one interfering target.

With this, as is apparent from (4.9.17), (4.9.19), optimum processing is reduced to subtracting signals (distinguished on a background of noise) from the target, and interference with corresponding coefficients. For brevity, such a method of processing is designated below as coherent compensation.

Considering in (4.10.16) m = 1, we obtain

$$q_1 = q_0\left[1 - \frac{q_п}{1+q_п}|C(\tau_1 - \tau, \Delta\omega_1)|^2\right]. \qquad (4.10.16')$$

where $q_п = \frac{z_1 T_э}{2N_0}$ is the interference-to-noise ratio.

The corresponding formula for periodic signal and fast fluctuations of interference

has the form

$$q_1 = q_0 \left[1 - |C_0(\tau_1 - \tau, \Delta\omega_Д)|^2 \times \right. \tag{4.10.17}$$

$$\left. \times \sum_{-\infty}^{\infty} \frac{\frac{\sigma_1}{2N_0} S_n(\Delta\omega_Д - k\Omega_r)}{1 + \frac{\sigma_1}{2N_0} S_n(\Delta\omega_Д - k\Omega_r)} \right].$$

As was possible to expect on the basis of the results of paragraph 4.9.2 and simply on the basis of purely qualitative reasonings, the signal-to-interference ratio in the output of the system of optimum processing is increased by the reduction in the correlation of the laws of modulation of the useful and interfering signals.

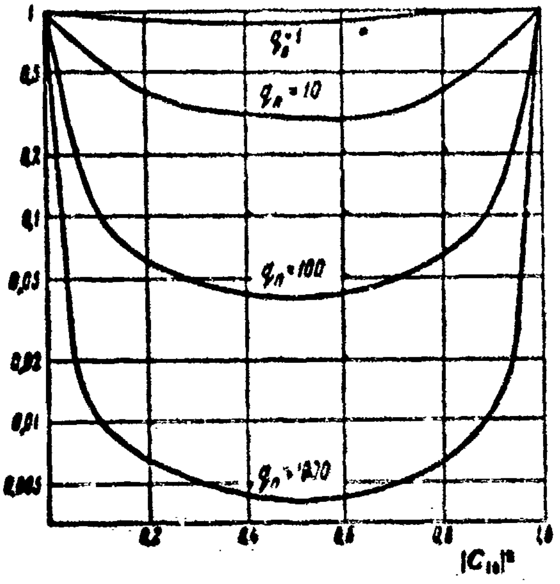

Fig. 4.21. Loss due to abandoning optimum processing during sharpened interference.

For the case of $Z_1(t, \tau) = u(t - \tau)$, by substituting $\sigma(x) = \sigma_1\delta(x - \tau_1)$ in (4.10.1) and (4.10.13), we obtain:

in the case of single transmission

$$q_1 = q_0 \frac{1}{1 + q_0 |C_0(\tau_1 - \tau, \Delta\omega_Д)|^2}; \tag{4.10.18}$$

in the case of periodic signal and fast fluctuation

$$q_1 = q_0 \frac{1}{1 + \frac{\sigma_1}{2N_0} |C_0(\tau_1 - \tau, \Delta\omega_Д)|^2 \sum_{-\infty}^{\infty} S_n(\Delta\omega_Д - k\Omega_r)}. \tag{4.10.19}$$

Comparison of formulas (4.10.18) and (4.10.16') shows that the use of optimum processing gives in the considered case, great practical gain, increasing infinitely

with the increase of intensity of interference: with the tendency of $q_{\text{п}}$ to infinity, q_1 for optimum processing, tends to

$$q_0[1-|C(\tau_1-\tau,\ \Delta\omega_{\text{Д}})|^2],$$

for $Z(t,\tau)=u(t-\tau)$ tends to zero.

Fig. 4.21 presents the dependence of relation (Γ) of magnitudes q_1 determined by formulas (4.10.16') and (4.10.18), characterizing the gain due to the use of optimum signal processing with various $q_{\text{п}}$, on $|C_{10}|^2=|C(\tau_1-\tau,\ \Delta\omega_{\text{Д}})|^2$. Gain is absent only during $C(\tau_1-\tau,\Delta\omega_{\text{Д}})=0;\ 1$ and maximum during $|C_{10}|^2=0{,}5$.

During fast fluctuation of interference and periodic signal, the dependence of Γ on $C_0(\tau_1-\tau,\ \Delta\omega_{\text{Д}})$ has the same form as that just considered. With this, the role of parameter $q_{\text{п}}$ is played by relation

$$\frac{e_1}{2N_0}\sum_{-\infty}^{\infty}S_{\text{п}}(\Delta\omega_{\text{Д}}-k\Omega_r).$$

In order to be certain of this, it is enough to remember that formula (4.10.17) is obtained on the assumption that $\Delta f_{\text{п}}T_r\ll 1$, so that

$$\sum_{-\infty}^{\infty}\frac{e_1S_{\text{п}}(\Delta\omega_{\text{Д}}-k\Omega_r)}{2N_0+e_1S_{\text{п}}(\Delta\omega_{\text{Д}}-k\Omega_r)}\approx$$

$$\approx\frac{e_1\sum\limits_{-\infty}^{\infty}S_{\text{п}}(\Delta\omega_{\text{Д}}-k\Omega_r)}{2N_0+e_1\sum\limits_{-\infty}^{\infty}S_{\text{п}}(\Delta\omega_{\text{Д}}-k\Omega_r)}.$$

Results obtained show that in a case of "discrete" interference, abandoning optimum processing is inexpedient, since such treatment allows us to ensure a sufficiently large signal-to-interference ratio even, when the usual method of processing with multiplication by the expected signal is completely weakened. The principle of coherent compensation of the signal from the interfering target, which is the main element of optimum processing, may be, apparently, successfully applied for a whole series of practical problems of distinguishing the signal from low-flying aircraft on a background of any local object, etc.

The main difficulty in the way of using this method, is the fact that in a number of cases the intensity and coordinates of the interference target are not known

accurately and are determined only on the basis of the radar signal. In this case, due to inaccurate knowledge of coordinates, the effectiveness of compensation decreases. On the intensity of the interfering target, the effectiveness of compensation during $q_{\text{п}} \gg 1$ depends only slightly, and this dependence can be disregarded.

In order to have the possibility of judging the degree of decrease in effectiveness, let us consider an example. Let the speed of the interfering target be known, and the distance be determined by the received signal with the help of the maximum likelihood method without calculation of the possible presence of a true target in the environments of the interfering target. With this, using expansion

$$C(x_0+\Delta x) \simeq C(x_0) + ia(x_0)\,\Delta x + \frac{b(x_0)}{2}(\Delta x)^2$$

and considering that b(0) is real, and a (0) with a symmetric spectrum of modulation is equal to zero; as a result of substitution of

$$Z_1(t,\tau) = \frac{1}{N_0}\,[u(t-\tau)\,e^{i\omega_1 t} - \\ - C(\tau_1' - \tau,\ \Delta\omega_д)\,u(t-\tau_1')\,e^{i\omega_1 t - i\Delta\omega_д(t-\tau_1')}]$$

(τ_1' is the measured value of delay of signal from changing target) in (4.10.1) we obtain

$$q_1 = q_0\,\frac{1-|C(\tau_1-\tau,\ \Delta\omega_д)|^2}{1+q_0(\Delta\tau)^2\,\dfrac{|a(\tau_1-\tau)|^2}{1-|C(\tau_1-\tau,\ \Delta\omega_д)|^2}}, \qquad (4.10.20)$$

where $\Delta\tau = \tau_1' - \tau_1$ is error in distance measurement.

In this error it is expedient to distinguish two components, one of which is caused by the influence of noises on the accuracy of measuring the distance of the interfering target, and the second is connected with the presence of the revealed target. As is shown in Chapter 7, Vol. II, the dispersion of fluctuating error during $q_{\text{п}} \gg 1$ is

$$\sigma_\tau^2 \approx \frac{1}{2q_0 b(0)}.$$

Systematic error can be determined by equating to zero the mean value of the logarithm of the relation of verisimilitude (4.3.9). Assuming this error $\Delta\tau_c$ to be small, we obtain

$$\Delta\tau_c \approx \frac{\dfrac{P_c}{\sigma_1}\,\mathrm{Re}\left[C(\tau-\tau_1)\,\dfrac{d}{d\tau}C^*(\tau-\tau_1)\right]}{b(0)+\dfrac{P_c}{\sigma_1}\{|a(\tau_1-\tau)|^2+\mathrm{Re}\,b(\tau-\tau_1)\,C^*(\tau-\tau_1)\}},$$

By substituting $\overline{\Delta\tau^2} = \sigma_\tau^2 + \Delta\tau_c^2$ in (4.10.20), it is simple to obtain a final expression for the signal-to-interference ratio q_1. It is interesting to note that $\frac{q_1}{q_0}$ with a decrease in intensity ratio of the true and interfering targets. This is connected with a decrease in systematic error with a decrease in $\frac{P_c}{q_1}$, due to which the accuracy of compensation of interference is increased.

Fig. 4.22 presents [calculated by formula (4.10.20)] dependence $\frac{q_1}{q_0}$ on $|C(\tau_1 - \tau)|^2$ for a case of sequence of Gaussian pulses with linear frequency modulation when $\Delta\omega_d = 0$, various $\frac{P_c}{q_1}$ and $q_0 = 200$ (this corresponds to $D = 0.9$ and $F = 10^{-10}$). For the considered law of modulation

$$C(\tau) = e^{-\frac{\pi^2\tau_u^2}{22.4}\tau^2};$$

$$a(\tau) = -\frac{(\pi\tau_u)^2}{11.2}\tau e^{-\frac{(\pi\tau_u)^2}{22.4}\tau^2};\ b(0) = \frac{(\pi\tau_u)^2}{11.2},$$

where τ_u is pulse duration based on a level of half power;

a is the speed of frequency change.

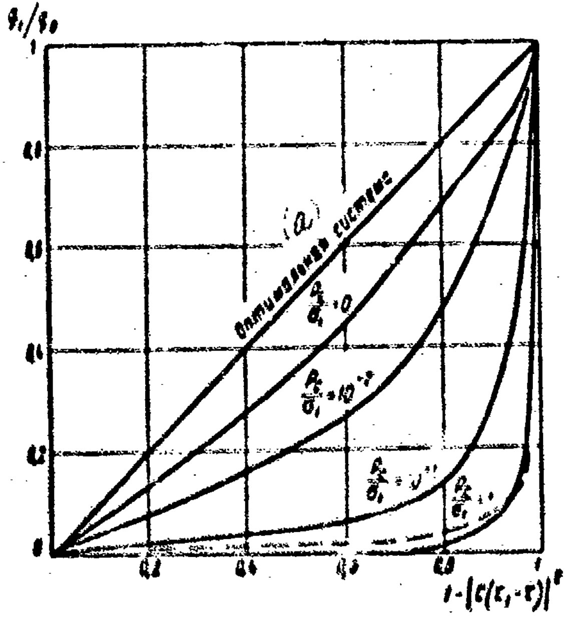

Fig. 4.22. Dependence of signal-to-interference ratio in the output of the processing system on the distance between targets.
KEY: (a) Optimum system.

For comparison on the same figure there is shown, by a dotted line, the dependence of $\frac{q_1}{q_0}$ on $|C(\tau - \tau_1)|^2$ for the case of $\frac{P_c}{q_1} = 1$, when as support, the expected

signal is used. As can be seen from the graph, in the case of different intensities of detectable and interfering targets, the shown processing method leads to almost the same results (very unsatisfactory) as coherent compensation of a signal from the interfering target, carried out in accordance with optimum processing. However, by measure of the decrease of $\frac{P_c}{\sigma_1}$ the effectiveness of compensation, both relatively and absolutely grows. For example, when $\frac{P_c}{\sigma_1}=10^{-2}$ and $|C(\tau_1-\tau)|^2=0,2$ detection on a background of an interfering target requires only a double increase of the threshold signal as compared with detection on a background of noise if we use coherent compensation, while in the usual correlated processing with multiplication by expected signal, detection in these conditions is practically impossible.

The considered example confirms the expendiency of the practical use of the coherent compensation method for distinguishing useful signals on a background of powerful interference reflections from false targets. When using the method, one should, however, consider that good results can be received only with accurate technical achievement of compensation. With this, the most substantial role is played by equipment error in the measurement of distance and speed and error in setting up the amplification factor of the compensation channel [see (4.9.19)]. Due to inevitable imperfection in fulfilling all these operations, growth of $\frac{q_1}{q_0}$ during decrease of $\frac{P_c}{\sigma_1}$, in practice will take place only up to a definite value of $\frac{P_c}{\sigma_1}$ the smaller, the bigger the indicated errors.

Let us consider a case of extensive interference. The signal-to-interference ratio q_1, corresponding to optimum processing of received signal, is calculated for this case by the formula (4.9.4) with use of results of paragraph 4.9.3.

Substituting (4.9.27) in (4.9.4), we obtain

$$q_1=\frac{P_c}{2N_0}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{|U_r(\omega)|^2\,d\omega}{1+\frac{S(\omega+\Delta\omega_1)}{2N_0}}. \qquad (4.10.21)$$

Spectral density $S(\omega)$ is determined in the case of single transmission by formula (4.9.25), and in the case of periodic signal by formula (4.9.26).

If the main signal is a complexly modulated single transmission with a nonperiodic autocorrelation function $C(\tau)$, then the spectral density of modulation $S_{\text{м}}(\omega)$, in (4.9.25), is a slowly changing function in comparison with $S_{\phi}(\tau, \omega)$. Therefore, disregarding, as before, the distortion of the modulation due to the Doppler shift of the spectral components and assuming in (4.9.25) $a=1$, and also considering that (see Section 1.2)

$$|U_{T}(\omega)|^2 = T_{\text{оф}} S_{\text{м}}(\omega),$$

it is possible to write the expression for q_1 in the form

$$q_1 = q_0 \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{S_{\text{м}}(\omega)\, d\omega}{1 + \frac{\sigma(\tau) T_{\text{эф}}}{2N_0} S_{\text{м}}(\omega - \Delta\omega_{\text{д}})}. \qquad (4.10.22)$$

Let us consider for comparison the signal-to-interference ratio for this case when using the expected signal as support. Considering in (4.10.1) $\sigma(x) S_{\phi}(x, \omega) \approx \approx \sigma(\tau) S_{\phi}(\tau, \omega)$ and $\Delta f_{\text{д}} \leqslant \Delta f_{\text{м}}$ ($\Delta f_{\text{м}}$ is the width of the spectrum of modulation), and also considering the property (1.2.7) of the function of uncertainty $|C(\tau, \omega)|^2$, we obtain

$$q_1 = q_0 \frac{1}{1 + \frac{\sigma(\tau) \Delta\tau_{\text{эф}}(\Delta\omega_{\text{д}}) T_{\text{эф}}}{2N_0}}, \qquad (4.10.23)$$

where

$$\Delta\tau_{\text{эф}}(\Omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} S_{\text{м}}(\omega + \Omega) S_{\text{м}}^{*}(\omega)\, d\omega. \qquad (4.10.24)$$

If, as this usually occurs, the Doppler shift $\Delta\omega_{\text{д}}$ is small as compared with the effective width of the modulation spectrum, then it is possible to replace $\Delta\tau_{\text{эф}}(\Delta\omega_{\text{д}})$ in (4.10.23) by $\Delta\tau_{\text{эф}}(0)$:

$$q_1 \approx q_0 \frac{1}{1 + \frac{\sigma(\tau) T_{\text{эф}}}{2N_0} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |S_{\text{м}}(\omega)|^2\, d\omega}. \qquad (4.10.23')$$

With these same conditions, it is possible to make a similar replacement in (4.10.22). Comparison of (4.10.23') and (4.10.22) shows that these expressions coincide with

rectangular spectral density

$$S_{\text{м}}(\omega)=\begin{cases}\frac{1}{\Delta f_{\text{м}}} & |\omega|<\pi\Delta f_{\text{м}},\\ 0 & |\omega|>\pi\Delta f_{\text{м}},\end{cases}$$

when, as one may see from (4.9.27), the optimum support signal coincides with the expected.

With this,

$$q_1=q_0\frac{1}{1+\frac{a(\tau)T_{\text{эф}}}{2N_0\Delta f_{\text{м}}}}. \qquad (4.10.25)$$

Signal-to-interference ratio q_1 increases by measure of the increase in effective width of the modulation spectrum $\Delta f_{\text{м}}$.

During less steeply drooping spectra of modulation the advantage of optimum processing becomes noticeable. For example, when

$$S_{\text{м}}(\omega)=\frac{1}{\Delta f_{\text{м}}}\frac{1}{1+\left(\frac{\omega}{2\Delta f_{\text{м}}}\right)^2}$$

the relation of magnitudes q_1, determined by formulas (4.10.23') and (4.10.22), is

$$\Gamma=\frac{\sqrt{1+\frac{a(\tau)T_{\text{эф}}}{2N_0\Delta f_{\text{м}}}}}{1+\frac{a(\tau)T_{\text{эф}}}{4N_0\Delta f_{\text{м}}}}=\frac{\sqrt{1+\gamma}}{1+\frac{1}{2}\gamma}.$$

Relation $\Gamma\approx 1$ when $\gamma<1$, $\Gamma\approx\frac{2}{\sqrt{\gamma}}$ when $\gamma\gg 1$, for example, $\Gamma\approx 0.2$ when $\gamma=100$. For a more rapidly drooping spectrum

$$S_{\text{м}}(\omega)=\frac{1}{\Delta f_{\text{м}}\left[1+\left(\frac{\omega}{4\Delta f_{\text{м}}}\right)^2\right]^2}.$$

Γ with increase of the interference-to-noise ratio γ diminishes more slowly:

$$\Gamma=\frac{\sqrt{1+\gamma}\sqrt{\sqrt{1+\gamma}-1}}{\sqrt{2}\left(1+\frac{5}{16}\gamma\right)};$$

when $\gamma=100$ magnitude $\Gamma\approx 0.7$, i.e., it is 3.5 times more than in the preceding example. When $\gamma\gg 1$ relation $\Gamma\approx\frac{2}{\sqrt{\gamma}}$.

Hence we can conclude that by measure of the increase in steepness of the droop of the spectrum, the loss due to the use of correlated processing with multiplication by the expected signal decreases.

The physical value of the regularity obtained consists of the following. As was shown, optimum processing of signal during extensive interference includes suppression in the expected signal from the target of the spectral components having the biggest intensity in the spectral density of the interference, which in broad terms coincides in form with the spectrum of modulation. With this, the spectrum of the support signal is compressed and expanded due to the relative increase in components outside the main maximum of the spectrum of modulation. The result of this is an increase in the quality of selection with respect to distance (resolving power with respect to distance) and simultaneously an increase of intensity of noise passing to the output of the receiver. The steeper the spectrum of modulation, droops, the greater increase of lateral components of the spectrum is required for a substantial increase in resolving power and the more noise increases. Consequently, with a given interference-to-noise ratio, the gain due to optimum processing of the signal will be less. A signal with rectangular spectral density of modulation does not have lateral components of the spectrum; therefore, the gain of optimum processing with such a signal is absent.

We will examine similarly a case of periodic modulation of the main signal. In this case, during high speeds of motion of a target and radar set, an essential role can be played by the difference in the Doppler shifts of certain spectral components of modulation; therefore, in the derivation of the corresponding formulas we will consider this effect. Assuming that $\Delta f_{\text{п}} T \gg 1$ and $T \gg T_r$, and substituting in (4.10.21)

$$|U_r(\omega)|^2 \approx \frac{2\pi T}{T_r^2} \sum_{k=-\infty}^{\infty} |U_0(k\Omega_r)|^2 \delta(\omega - k\alpha_1\Omega_r) \qquad (4.10.26)$$

and in (4.9.26), we obtain

$$q_1 \approx q_0 \sum_{k=-\infty}^{\infty} \frac{\frac{1}{T_r} S_{\text{м}0}(k\Omega_r)}{1 + \frac{\sigma_0(\tau)}{2N_0} \sum_{l=-\infty}^{\infty} S_{\text{м}0}(l\Omega_r) S_{\Phi 0}[\tau_1 \Delta\omega_{\text{д}} + (\alpha_1 k - \alpha l)\Omega_r]} \qquad (4.10.27)$$

where $\alpha_1 = 1 + \frac{2v_r}{c}$,

$v_{\text{ц}}$ is the speed of target motion relative to the radar set.

The magnitude of the signal-to-interference ratio essentially depends on Doppler shift $\Delta\omega_{\text{д}}$ and is greatest during maximum noncoincidence of the harmonics of the useful signal with the harmonics of the interference. During $a \approx a_1 \approx 1$ maximum q_1 occurs when $\Delta\omega_{\text{д}} = \left(l + \frac{1}{2}\right)\Omega_r$ (l is an integer), and minimum — when $\Delta\omega_{\text{д}} = l\Omega_r$. Speeds of targets, corresponding to minimum q_1, usually are called blind. When $\Delta f_{\text{п}} \ll \frac{\Omega_r}{2\pi}$ the gap between maximum and minimum is very large. With blind speeds of a target, its detection on a background of passive interferences in actual conditions is frequently impossible.

Dependence of q_1 on $\Delta\omega_{\text{д}}$ becomes less sharp and the effectiveness of frequency selection decreases, if the speeds of the target and the passive interference relative to the radar set differ so strongly that the difference in Doppler shifts of the spectral components becomes essential. Physically, this weakening of dependence is explained by the fact that during the fast motion of the target relative to the reflectors forming the interference, together with the signal from the target in various periods, signals are selected from various reflecting elements of passive interference, which are not correlated with each other. Due to this, the equivalent spectral density of interference on input of the narrow-band filter expands. When the shift of target relative to the reflectors of interference for the period of repetition exceeds the magnitude of the resolution range with respect to distance, interference in input of the filter by its properties approaches white noise and dependence on $\Delta\omega_{\text{д}}$ is practically absent.

These phenomena can be described, using an exclusively spectral concept and formula (4.10.27).

When $\Delta f_{\text{п}} \ll \frac{\Omega_r}{2\pi}$ in all, in the denominator (4.10.27), an essential role is played by one or two members with $l \approx \frac{a_1}{a} k - \frac{\Delta\omega_{\text{д}}}{\Omega_r}$.

If $a_1 \approx a$, then during all k maximum $S_{\Phi 0}[\omega_{\text{д}} + a(k - l)\Omega_r]$, as function l, has the same magnitude. When $a_1 \neq a$ the magnitude of the maximum depends on k and with

a change of k from zero to $k_1 = \frac{\pi\Delta f_{\text{и}}}{\Omega_r} \approx k_{\max}$ it can change very significantly. If $\left|\frac{a_1}{a} - 1\right| \frac{\pi\Delta f_{\text{и}}}{\Omega_r} \gg 1$, then with a change of k from zero to k_1 the indicated maximum succeeds many times in acquiring all possible values and as a result, q_1 weakly depends on $\Delta\omega_{\text{д}}$. Thus, the degree of influence of the difference in the Doppler shifts of separate spectral components is characterized by magnitude

$$\delta = \left|\frac{a_1}{a} - 1\right| \frac{\pi\Delta f_{\text{и}}}{\Omega_r} \approx |v_{\text{ц}} - v_{\text{п}}| \frac{2\pi\Delta f_{\text{и}}}{c\Omega_r} = \frac{|v_{\text{ц}} - v_{\text{п}}|\, T_r}{2\Delta d}, \tag{4.10.28}$$

where $\Delta d = \frac{c}{2\Delta f_{\text{и}}}$ is the extent of the resolution range with respect to distance.

From (4.10.28) it is clear that the degree of influence of the considered effect is determined by the magnitude of the ratio of target displacement, relative to passive interference for a period, to the length of the resolution range Δd. With the values of parameters typical for contemporary radar, δ is quite small. Magnitude Δd is limited from below by dimensions of the target and $\min \Delta d \approx 10$ м. With this value of Δd and $|v_{\text{ц}} - v_{\text{п}}| \sim 10^2 \div 10^3$ m/sec, it is sufficient to take a repetition frequency on the order of 1 — 10 kilocycles, so that δ is less than one. In accordance with this, it is possible to put in (4.10.27) $a_1 = a = 1$ and, disregarding shift of spectrum $S_{\text{и}0}(\omega)$ to magnitude $\Delta\omega_{\text{д}}$, copy this formula in the form

$$q_1 \approx q_0 \sum_{k=-\infty}^{\infty} \frac{1}{T_r} \frac{S_{\text{и}0}(k\Omega_r)}{1 + \frac{\sigma_0(\tau)}{2N_0} S_{\text{и}0}(k\Omega_r) \sum\limits_{l=-\infty}^{\infty} S_{\Phi 0}(\tau, \Delta\omega_{\text{д}} - l\Omega_r)} \approx \tag{4.10.29}$$

$$\approx q_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_{\text{и}0}(\omega)\, d\omega}{1 + \frac{\sigma_0(\tau)}{2N_0} S_{\text{и}0}(\omega) \sum\limits_{-\infty}^{\infty} S_{\Phi 0}(\tau, \Delta\omega_{\text{д}} - l\Omega_r)}.$$

Similarly to (4.10.13) for the case when we used the expected signal as a support signal, we obtain

$$q_1 = q_0 \frac{1}{1 + \frac{\sigma_0(\tau)}{2N_0} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |S_{\text{и}0}(\omega)|^2\, d\omega \sum\limits_{k=-\infty}^{\infty} S_{\Phi 0}(\tau, \Delta\omega_{\text{д}} - k\Omega_r)}. \tag{4.10.30}$$

Formulas (4.10.29) and (4.10.30) in form coincide with (4.10.22) and (4.10.23'),

respectively. Instead of magnitude $\sigma(\tau)T_{\text{эф}}$ in (4.10.29) and (4.10.30) there is contained

$$\sigma_{\text{э}}(\tau)\sum_{-\infty}^{\infty} S_{\Phi_0}(\tau, \Delta\omega_{\text{д}} - k\Omega_r).$$

In accordance with this, the results (received above for a case of single transmission) of comparing optimum processing with correlated processing, including multiplication by the expected signal, can be extended to a case of periodic modulation.

Thus, in cases of single transmission and a periodic main signal when $\Delta f_{\text{и}}T \gg 1$ the result of comparing the considered methods of signal processing depends on the form of spectral density of modulation. Difference in effectiveness increases by measure of decrease in the steepness of droop $S_x(\omega)[S_{x0}(\omega)]$ and is absent with rectangular spectral density of modualtion, when the optimum support signal coincides with that expected, and optimum processing is discrete. Near such spectral density is, for example, function $S_x(\omega)$ for a signal with linear frequency modulation.

4.10.3. Selection of a Law of Modulation of the Main Signal from the Viewpoint of Noise-Resistance in Reference to Passive Interferences

Inasmuch as the properties of a signal reflected from a passive interference to a significant degree, are determined by the form and parameters of the law of modulation used, it is natural to expect that noise-resistance in reference to this form of interference can be increased by means of a rational selection of a law of modulation. In this division on the basis of analysis of the expressions obtained above for the signal-to-interference ratio q_1, determining the character of the dependence of detection reliability on the form of modulation, requirements will be formulated, which the law of modulation should satisfy guaranteeing relatively high noise-resistance in reference to passive interference*. These requirements

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*Selection of a concrete law of modulation is usually made from a class of laws, possessing approximately identical properties, and is determined basically by technical considerations.

are different for cases of discrete and extensive interference, for which reason it is expedient, as before, to consider these cases separately.

In the case of discrete interference, noise-resistance is determined by the value of the function of uncertainty with values of $\Delta\tau$ and $\Delta\omega_{\text{д}}$, characterizing the detuning of the target and interference with respect to delay and frequency. The lower this value, the better the target is selected on a background of interference. Inasmuch as target and interference position data usually are unknown beforehand, it is desirable that $|C(\Delta\tau, \Delta\omega_{\text{д}})|^2$ be sufficiently small during all prior possible values of $\Delta\tau$ and $\Delta\omega_{\text{д}}$. The best from this point of view would be a signal, for which the function of uncertainty is near zero with all values of τ, Ω outside the main maximum $(\tau=0,\ \Omega=0)$. However, the possibilities of creating such signals are limited by the integral properties of the function of uncertainty (1.2.10), (1.2.7), and (1.2.9) considered in Section 1.2.

In accordance with property (1.2.10), the total volume, included between plane (τ, Ω) and relief of function $|C(\tau, \Omega)|^2$, is constant. Thus, if coordinates τ and Ω of interference targets fill plane (τ, Ω) sufficiently densely and evenly, then it is impossible to increase noise-resistance, due to rational selection of a form of main signal (in any case, during the use of a correlated method of processing by multiplication by the expected signal).

From (1.2.7) it follows that the sectional area of relief of function $|C(\tau, \Omega)|^2$ along axis τ with an increase of $|\Omega|$ diminishes slowly if $|C(\tau)|^2$ has one maximum on axis τ and quickly diminishes outside this maximum. This area, which it is possible to consider also as effective width $\Delta\tau_{\text{эф}}(\Omega)$ of the function of uncertainty with respect to τ, diminishes rapidly only when the spectrum of modulation has a lined structure. With this, function $C(\tau)$ necessarily has secondary maximums caused by the presence of ambiguity with respect to distance.

Finally, property (1.2.9) determines the speed of decrease in the sectional area of relief $|C(\tau, \Omega)|^2$ with respect to Ω with growth $|\tau|$, which it is possible to

consider as effective width $\Delta\Omega_{\text{эф}}(\tau)$ of the function of uncertainty with respect to variable Ω. Function $\Delta\Omega_{\text{эф}}(\tau)$ quickly diminishes only in the case of pulse radiation, since phase modulation does not affect the character of change $\Delta_{\text{эф}}\Omega(\tau)$.

From the shown properties it follows that if we want to ensure equally good resolving power on all the plane (τ, Ω), then the best achievement on this path would be the obtaining of a function of uncertainty available outside the limits of the main maximum when $|\tau| < \frac{T}{2}$ (T is the duration of signal) and $|\Omega| < \pi\Delta f_{\text{м}}$ ($\Delta f_{\text{м}}$ is the effective width of the spectrum of modulation) constant level, approximately equal to $\frac{1}{\Delta f_{\text{м}} T}$. To decrease the level of function $|C(\tau, \Omega)|^2$ outside the limits of the main maximum is possible only on a limited section of plane (τ, Ω) by means of a redistribution of values of $|C(\tau, \Omega)|^2$. Such redistribution occurs, for example, during the use of a periodic signal possessing ambiguity with respect to distance and speed, but then ensuring a very low level of $|C(\tau, \Omega)|^2$ in the intervals between the secondary maximums (see Section 1.2).

The influence of relatively small secondary maximums of the function of uncertainty can be removed with the use of methods of signal processing close to those considered in paragraph 4.9.2. Analysis of possibilities appearing in connection with this will be conducted in Chapter 13 which is devoted to the problem of resolving power.

In a case of extensive interference, the signal-to-interference ratio does not depend, as one may see from corresponding formulas of paragraph 4.10.3, on the form of the function of uncertainty, but is determined by spectrum $S_{\text{м}}(\omega)$ of function $C(\tau)$. The disappearance of dependence on $C(\tau, \Omega)$ is caused, as the transition from (4.10.1) to (4.10.23) has shown, by the integral property (1.2.7) of the function of uncertainty, in accordance with which magnitude $\Delta\tau_{\text{эф}}(\Omega)$, representing, as it were, the equivalent extent of pulse volume [see (4.10.23)] is determined only by the spectral density of modulation $S_{\text{м}}(\omega)$.

If function $C(\tau)$ has the form of a short pulse, then the spectrum $S_{\text{м}}(\omega)$ is wide

and its displacement, due to the Doppler shift of the target frequency relative to the interference, can be disregarded. With this, selection of a target on a background of passive interferences with respect to speed is absent and only selection with respect to distance can be used. Selection of a signal should be made in this case based on conditions of maximum

$$q_1 = q_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_{\text{м}}(\omega)\, d\omega}{1 + \frac{\sigma(\tau) T_{\text{эф}}}{2N_0} S_{\text{м}}(\omega)} .$$

It is possible to show that with a limited frequency band reserved for a given radar set, the maximum of q_1 occurs in rectangular spectral density of modulation. However, the optimum nature of such a signal is very relative. If we do not require equality of $S_{\text{м}}(\omega)$ to zero outside a given frequency band, and consider the effective band of modulation fixed, then a signal with rectangular spectral density yields to many others, including a signal with Gaussian spectral density, although, in general, the difference between signals of various forms with the same effective band $\Delta f_{\text{э}}$ is not substantial. We can make certain of this by comparing values $\Delta\tau_{\text{эф}}$. corresponding to various $S_{\text{м}}(\omega)$. with the same effective band.

For all signals of such form, the signal-to-interference ratio is determined approximately by formula (4.10.25). With large values of the interference-to-noise ratio, the unit in the denominator of this formula can be disregarded. With this,

$$q_1 \approx \frac{s_{\text{ц}}}{s_{\text{п}}} ,$$

where $s_{\text{ц}}$ is the effective reflecting surface of the target;

$s_{\text{п}}$ is the effective reflecting surface of the interference, occurring in the effective resolution range with respect to distance $\frac{c}{2}\ \Delta\tau_{\text{эф}}$, located near the target.

This signal-to-interference ratio even with a resolving power of the order of the dimensions of the target can constitute $10^{-2} - 10^{-3}$. Detection of the target with noticeable probability with such a signal-to-interference ratio is practically impossible. Hence we can conclude that to ensure high noise-resistance without the

use of selection by speed is impossible.

Certain assumptions used in obtaining this result may cause one essential objection. We assume that properties of the interference in the environment of the target do not change in a range outside which function $|C(\tau, \Omega)|^2$ is practically equal to zero. However, as was noted above, it can appear that function $|C(\tau, \Omega)|^2$ outside the limits of the main maximum has an approximately constant level in the delay interval, an approximately equal duration of signal, even significantly exceeding the extent of interference. In this case, this assumption is not fulfilled and for an estimate of noise-resistance additional calculation is necessary. For a case of correlated processing with multiplication by the expected signal from (4.10.1), considering $|C(x-\tau, \Delta\omega_{\text{д}})|^2 \approx \frac{1}{\Delta f_{\text{м}} T}$ and disregarding noise as compared with interference, we obtain

$$q_1 \approx \frac{s_{\text{ц}}}{s_{\text{п}} \frac{\Delta\tau_{\text{п}}}{\Delta f_{\text{м}} T \Delta\tau_{\text{эф}}}} \approx \frac{s_{\text{ц}}}{s_{\text{п}}} \frac{T}{\Delta\tau_{\text{п}}},$$

where $\frac{c}{2}\Delta\tau_{\text{п}}$ is the extent of interference with respect to distance;

$s_{\text{п}}$ as before, is the reflecting surface of reflectors forming the passive interference and occurring in the volume of resolution.

The signal-to-interference ratio obtained differs from the above considered by $\frac{T}{\Delta\tau_{\text{п}}}$ times. If $\frac{s_{\text{ц}}}{s_{\text{п}}} = 10^{-2} - 10^{-3}$, then to obtain $q_1 = 100$, corresponding (see Section 4.4), during slow fluctuation of target, to probabilities of correct detection $D = 0.9$ and false alarm $F = 10^{-4}$, it is necessary that ratio $\frac{T}{\Delta\tau_{\text{п}}}$ is equal to $10^4 - 10^5$. The observation time of the signal from any fixed direction for a radar set in the conditions of the survey does not exceed 0.03 — 0.1 sec. Hence it follows that the required signal-to-interference ratio can be obtained only at $\Delta\tau_{\text{п}} \approx$ $\approx 0.3 - 10$ microseconds (50 — 1,500 m in distance). At the same time in real conditions, the extent of interference with respect to distance can be significantly more (up to several tens of kilometers).

Thus, the calculation conducted also confirms the conclusion made that without the use of frequency selection we cannot ensure sufficient noise-resistance in

reference to passive interference, in certain important cases, with any law of modulation of the main signal. Such a selection in combination with resolving power with respect to distance is possible, as examination has shown, only with a line spectrum of the law of modulation of the main signal. Only lined structure of spectrum allows us to obtain a fast decrease of function $\Delta\tau_{\text{эф}}(\Omega)$ with an increase of $|\Omega|$.

The most widespread signals of such form are periodic signals with arbitrarily intraperiodic modulation. However, a high ambiguity with respect to distance is peculiar to these signals. The majority of radar stations, by selection of repetition frequency, do not manage to ensure simple selection with respect to speed and distance simultaneously in all the prior interval of change of these parameters.

In connection with this, it is tempting to form a nonperiodic signal with a line spectrum, ensuring simple measurement of large distances with sufficiently large distances between spectral lines. For measurement of distance with such a signal, as has been repeatedly suggested [65], we can use the differences in phases of certain spectral components of the reflected signal, and for simple measurement of great distances it is possible to use differences of the highest orders (the differences of phase differences).

We will calculate the functions of uncertainty $|C(\tau,\Omega)|^2$ and magnitude $\Delta\tau_{\text{эф}}(\Omega)$ for such a signal. Designating by A_j and ω_j amplitude and frequency of the j-th spectral component, we obtain

$$C(\tau,\Omega)=\frac{1}{T}\int\limits_0^T\sum_{j,k=1}^{n}A_jA_k^*\,e^{i(\omega_j-\omega_k)t+i\omega_j\tau+i\Omega t}\,dt, \tag{4.10.31}$$

where n is the number of spectral components.

In (4.10.31) it is assumed that $\sum\limits_{j=1}^{n}|A_j|^2=1$. If $|\omega_j-\omega_k|T\gg1$ and $|\Omega|\ll|\omega_j-\omega_k|$ during all $j\neq k$, which should be fulfilled for good selection with respect to frequency, then

$$|C(\tau,\Omega)|^2=\frac{\sin^2\frac{\Omega T}{2}}{\left(\frac{\Omega T}{2}\right)^2}\sum_{j,k=1}^{n}|A_jA_k|^2\cos(\omega_j-\omega_k)\tau=\frac{\sin^2\frac{\Omega T}{2}}{\left(\frac{\Omega T}{2}\right)^2}|C(\tau)|^2. \tag{4.10.32}$$

Substituting (4.10.32) in (1.2.7), we obtain

$$\Delta\tau_{\text{эф}}(\Omega)=\frac{\sin^2\frac{\Omega T}{2}}{\left(\frac{\Omega T}{2}\right)^2}\Delta\tau_{\text{эф}}=\frac{\sin^2\frac{\Omega T}{2}}{\left(\frac{\Omega T}{2}\right)^2}T\sum_{j=1}^{n}|A_j|^4. \qquad (4.10.33)$$

In particular, with equal distribution of power between spectral components $\left(|A_j|^2=\frac{1}{n}\right)$

$$\Delta\tau_{\text{эф}}=\frac{T}{n}.$$

Thus, to decrease the resolution range with respect to distance $\Delta\tau_{\text{эф}}$ and to increase the resolving power, it is necessary to increase the number of frequencies used. The same conclusion is confirmed and presented in Fig. 4.23 by dependence $|C(\tau)|^2$ for case n = 3 and $|A_j|^2=\frac{1}{3}$ $(j=1,2,3)$. As can be seen from the figure, with a small number of spectral components the function of uncertainty has large secondary maximums.

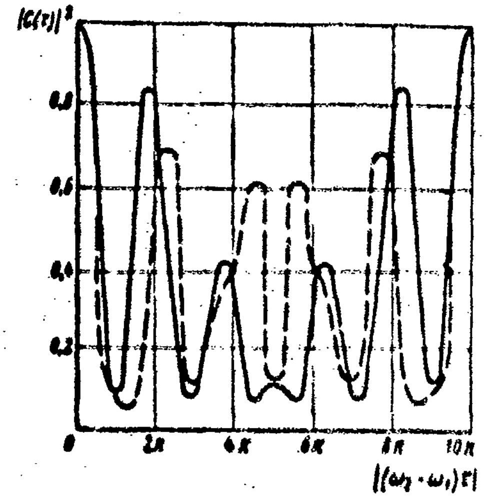

Fig. 4.23. Function of uncertainty $|C(\tau)|^2$ for a signal with line spectrum when n = 3. $\left(n=\left|\frac{\omega_2-\omega_1}{\omega_3-\omega_1}\right|.\right)$
——— n = 0.8, — — — — n = 1.2.

Of significant interest is the question whether an increase in spectral components must be accompanied by approximation of a formed signal in its properties to the usually utilized periodic signals, or whether a signal with significantly

higher selective properties can be obtained. Based on physical considerations, and also considering the available results of the theory of almost periodic functions, it is possible to doubt the existence of such a possibility; however, certain progress along this path may be, apparently, attained, and the solution of this problem deserves attention. At present the only signals accessible for practical use, with line spectrum and high resolving power with respect to distance are periodic signals possessing the advantage that their obtaining will be considerably less difficult technically, than obtaining signals formed from separate sinusoids. Therefore, during the following presentation we will orient ourselves in this form of main signals.

If we assume that the signal should be periodic, then for a final determination of its properties it remains to select the form of intraperiodic modulation, repetition frequency and resolving power with respect to distance, determined by the effective width of the spectrum of modulation. From intraperiodic modulation there is required only a sufficiently fast decrease of function $|C_0(\tau)|^2$ and the absence of secondary maximums for this function. As was shown in Section 1.2, sufficiently good properties from this point of view are possessed, for example, by signals with phase-code manipulation and certain pulse signals with linear frequency modulation; regarding the advantages of one of these forms of modulation, if there are such, they are purely technical and are connected with the realization of a modulator and receiver.

The greatest difficulties arise during selection of the repetition frequency. With this, in most cases it is necessary to choose between ambiguity in distance and speed since we cannot ensure simple measurement of both parameters simultaneously. Very frequently, a problem of simple measurement of speed is not set up and ambiguity in distance is introduced only because with low repetition frequency determined by maximum distance, the correlation of interference in neighboring periods turns out to be weak and the quality of frequency selection is lowered. This question will

be considered more specifically in the following paragraph. Sometimes ambiguity can be removed by using certain additional measures, part of which we shall now consider briefly.

The simplest method of removing ambiguity in distance is to consider the target at the moment of detection to be in the remote zone of ambiguity of the number of zones in the range of the radar. This method will apply, obviously, if the number shown is small.

For removal of ambiguity both in distance and in speed, we can use wobbling of the repetition frequency [68], which must be carried out fairly slowly in order not to affect the quality of selection with respect to speed. Let us assume, for example that to remove ambiguity in distance two changeable periods of repetition T_{r1} and T_{r2} are used. True delay τ is connected with delays τ_1' and τ_2' relative to the beginning of the period, by relationships

$$\tau = k_1 T_{r1} + \tau_1' = k_2 T_{r2} + \tau_2'. \tag{4.10.34}$$

If

$$\tau_{max} < \frac{T_{r1} T_{r2}}{|T_{r1} - T_{r2}|},$$

then when $T_{r1} > T_{r2}$ one of the following equalities is necessarily fulfilled:

$$k_2 = k_1, \quad \text{if} \quad \tau_1' > \tau_2',$$

$$k_2 = k_1 + 1, \quad \text{if} \quad \tau_1' < \tau_2'.$$

Using these relationships, it is simple to exclude k_1 and k_2 from (4.10.34) and to find τ. The essential deficiency of this method consists of the fact that in the presence of two and more targets insolvable with respect to speed, we cannot eliminate ambiguity because of the impossibility of identifying the pair of values τ_1' and τ_2'. Significantly more promising from this point of view is the method of smooth change of period with the simultaneous tracking of detected target with respect to distance. For example, if the period changes linearly, then the identification marks of the targets, being in the first interval of ambiguity from the radar, are motionless; in the second interval they move with the same speed, with which T_r changes, and in the k-th interval with a speed (k — 1) times greater. Measuring the speed

of motion of the identification mark of the revealed target, it is possible, thus, to remove ambiguity with respect to distance.

The same method, obviously, will apply for removal of ambiguity in speed. If the signal from the target occurs in the narrow-band filter of the system of detection* by the k-th harmonic, then during a change of repetition frequency, the frequency of this harmonic changes with a speed k times greater than the speed of change of the frequency of movement. The law of change of period T_r should, certainly, be chosen to exclude the effect of target motion.

In certain single-purpose radars, taking into account the possibility of the presence of passive interference, it is expedient to use a high repetition frequency ensuring a simple lock-on with respect to speed. With this, there frequently appears one more difficulty, connected with the impossibility of using spaced transmitting and receiving antennas, and consequently, with the impossibility of ensuring the required by-passing of receiver from the transmitter without cutoff of receiver for the time of radiation. This leads to the necessity of using a pulse signal and to the appearance of blind zones with respect to distance. A signal from targets in blind zones arrives when the receiver is closed.

To remove the effect of blind zones and simultaneously to eliminate ambiguity, the search for target and its subsequent range tracking can be accomplished by means of change of repetition frequency. In tracking conditions, this change must be made in such a manner that the signal from the target is always in a fixed point of the period. With this, as it is easy to check, distance can be calculated by formula

$$d(t) = -\frac{2T_r(t)}{\dot{T}_r(t)}v(t),$$

where $T_r(t)$ and $\dot{T}_r(t)$ are the current values of the period and its derivative;

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*In the presence of ambiguity, the block of filters should, obviously, cover only a range of frequencies, equal to the frequency of repetition, and the channels with respect to distance in this case should cover only one period.

$v(t)$ is the current value of speed measured by a system of tracking with respect to speed.

In radar systems, before which a problem of simple measurement of speed is not placed, but only the reliable selection of a target on a background of passive interference is required, a change of the frequency of repetition in the process of work, and also simultaneous work on several carrier frequencies can be used to remove the effect of blind speeds in the working speed range. These methods of combatting blind speeds will be examined in the next paragraph during the investigation of methods of separate processing of discrete periods of the signal.

4.10.4 Concerning the Question of Selecting Resolving Power* with Respect to Distance

Till now in this chapter, the target always has been assumed to be a point (small, as compared to the extent of the resolution range with respect to distance). With this, for extensive interference, noise-resistance always increases during expansion of the spectrum of modulation, i.e., with an increase of resolving power with respect to distance. During decrease of resolution range Δd it sooner or later becomes comparable with the dimensions of target l and the idealizations taken cease to be fulfilled. The question, arises, whether to increase further the resolving power.

In order to answer this question accurately, it is necessary to solve accurately the problem about optimum processing of a signal from an extensive target. With this, one can hardly be limited to investigation of a model of a target in the form of a combination of brilliant points, since in the theory of radiovision, transition to which occurs when $\Delta d \ll l$, an essential role should be played by the detailed study and comparison of "thin" structures of reflected signals from various targets for manifestation of their similarity and differences. With this, there appears the

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*Here, in essence, is considered resolving power, ensured during correlated processing with multiplication by expected signal. More specifically, the problem of resolution of targets will be considered in Chapter 13.

problem about the scattering of modulated electromagnetic waves on objects with dimensions of heterogeneities, comparable with wave length, corresponding to the difference of extreme frequencies of modulation. Solution of these problems is so far a matter of the future. Therefore, here we will be limited to discussion of the question about selection of resolving power from the viewpoint of the contemporary level of development of radar technology and contemporary ideas in this region.

Let us consider first of all the question about the consequences of decrease in resolution range to l and below, from the viewpoint of fulfillment by the radar set of its basic functions in the absence of interference. To these functions belong: the detection of a target or a group of targets and the determination of the parameters of the trajectories of the target. For idealization of target in the form of a combination of brilliant points, the quality of fulfillment of these functions can be investigated quite strictly.

Analysis of detection conditions can be conducted with the help of the results of paragraph 4.9.1. The function of correlation of the signal from the target is determined for an extensive target by formula (4.9.1), and function $V(t_1, t_2)$, in (4.2.2), can be written in the form

$$V(t_1, t_2) = \frac{1}{N_0}\delta(t_1 - t_2) - W_{\text{н}}(t_1, t_2)$$

where $W_{\text{н}}(t_1, t_2)$ is determined by equality (4.9.8). We will assume now that $\sigma(x)$ and $r(x, t)$ change slowly as compared with $C(x)$. Then it is possible to convert both parts of equation (4.9.11) according to Fourier witn respect to $x_1 - x_2$, considering x_1 constant. As a result we obtain

$$w_1(x_1, s, t_1, t_2) + \frac{\sigma(x_1)}{2N_0}\int_0^T w_1(x_1, s, t_1, t)\, S_{\text{м}}(s)\, r(x_1, t - t_2)\, dt = \frac{1}{N_0^2}\sigma(x_1)\, r(x_1, t_1 - t_2), \qquad (4.10.35)$$

where w_1 is the result of Fourier transform from $w(x_1, x_2, t_1, t_2)$ with respect to $x_1 - x_2$.

Equation (4.10.35) can be solved relative to w_1 as functions of time for extreme

cases of slow and fast fluctuation by methods repeatedly used earlier. For brevity, here we will limit ourselves to a case of slow fluctuation. With this, $w_1(x_1, s_1, t_1, t_2)$ does not depend on time and

$$w(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma(x_1)\, e^{is(x_1 - x_2)}}{N_0^2 \left(1 + \frac{\sigma(x_1)}{2N_0} S_M(s)\right)} ds. \qquad (4.10.36)$$

Substituting this solution in (4.9.8) and then in the expression for $V(t_1, t_2)$, relationships can be obtained, determining the character of optimum operations, which in general form are very complicated. It is possible to simplify these operations by approximating $S_M(s)$ in (4.10.36) a П-shaped curve. Then

$$V(t_1, t_2) \sim \operatorname{Re} \int_0^{\infty} \frac{\sigma(x)\, u(t_1 - x)\, u^*(t_2 - x)\, dx}{1 + \frac{\sigma(x) T}{2N_0 \Delta f_M}} e^{i\omega_0 (t_1 - t_2)}. \qquad (4.10.37)$$

From (4.10.37) it follows that optimum processing of a signal from an extensive target during the assumptions made includes optimum processing of signals from separate points of the target and the integration of results of processing with respect to delay, with weight depending on the distribution of the reflecting surface with respect to the extent of target. By approximation, it is possible to carry out this processing with the help of a finite number of channels, detuned by distance approximately by magnitude Δd. The output of the channels should be stored incoherently. Signals receivable on each of the channels, for the considered model, can be considered statistically independent. In this case, the characteristics of detection are precisely the same as in the case of the simultaneous use of the frequency channels considered in Section 4.6. From analysis of the characteristics, it follows that during slow fluctuation, optimum magnitude of resolution range exists:

$$\Delta d \approx \frac{l}{1 + [\Phi^{-1}(D)]^2},$$

and during fast fluctuation, the quality of detection is lowered with a decrease of Δd (threshold signal is calculated approximately as $\sqrt{\frac{l}{\Delta d}}$).

Thus, if conditions of statistical independence of signals reflected from separate sections of a target are executed, and the target fluctuates slowly, an increase

of resolving power up to a definite limit is meaningful even after Δd comes up to $\underline{l}$. The fact that this gain coincides with the gain due to the use of several frequency channels, allows us to choose between these two methods of increasing free-space range. At present, from technical considerations, preference should, apparently, be given to multifrequency radiation.

During measurement of coordinates of an extensive target, the question arises as to what is implied by measured coordinates. Apparently, it is expedient to speak of the coordiantes of the "radar center of gravity" of the target. If signals from separate sections of the target fluctuate independently, then the center strays in a random manner along the surface of the target and at any resolving power errors of determination of true target position cannot be significantly less than the dimensions of the target. This conclusion is completely confirmed by analysis of the accuracy of distance measurement conducted in Chapter 7, Vol. II. Thus, from the viewpoint of increase in accuracy of measurement of coordinates, the discussed increase of resolving power is not meaningful.

We will observe how the increase of resolving power affects noise-resistance in reference to passive interferences. An increase in the signal-to-interference ratio during expansion of the modulation band which takes place because of decrease in reflecting surface of the interference occurring in the resolution range in respect to distance, will, obviously, continue only until $\Delta d > l$ and then this growth will stop or, in any case, will be delayed (the specific form of dependence is determined by the properties of the target and the interference). At the same time, during further increase of resolving power (see paragraph 4.10.2) conditions worsen for selection of a moving target with respect to speed.

All the above mentioned arguments against increasing the resolving power after the resolution range becomes comparable with the dimensions of the target are based on contemporary ideas about the problems and possibilities of radar. Maybe in the future we will manage to apply increased resolving power (radiovision) for identification of targets and for singling out targets on a background of passive

interferences. During resolution of these problems, correlation of signals reflected from separate sections of the target should, apparently be used. The study of the problems connected with radiovision is only now starting, and it is impossible to predict all the difficulties which will arise and all the advantages which we will manage to put into use. However, it is possible to affirm that transition to a fuller use of a fine structure of signal will require multiple increase in the signal-to-noise ratio, since the quantity of information transmitted along the radar channel will increase. In connection with this, in systems of remote detection, it is possible that in the future resolving power on the order of the dimensions of the target will be used.

4.11. Separate Processing of Periods of Received Signal in the Presence of Passive Interferences

Analysis of the dependence of detection quality on the used law of modulation of the main signal has shown that effective selection of a target on a background of powerful, extensive, passive interference is possible virtually only during line spectrum of modulation, where at present only periodic signals are available for use from signals of this class which ensure the very high resolving power with respect to distance that is usually required. In connection with this, it is natural to give special attention to investigating the possibilities of replacing very complicated optimum operations by simpler ones, namely, for that form of signals.

As has already been noted, optimum processing of the received signal in the presence of extensive passive interference and noises is not broken down, generally, into intra- and interperiodic signals. In connection with this, for this processing we can not use reducing filters designed for the processing of separate periods of signal; processing becomes considerably different than that utilized in the absence of interference and all the system of detection built on optimum principles becomes very complicated. Therefore, of essential interest is the investigation of possibilities of using separate processing of periods in the presence of passive

interferences, to which this paragraph is devoted. With this, in accordance with the general aim of this book, we will find the optimum methods of intra- and interperiodic processing, assuming a separate character of processing from those given beforehand. Along with optimum methods, we will consider practical methods of interperiodic processing, which is reduced to period-by-period subtraction of any multiplicity of results of intraperiodic processing and subsequent incoherent accumulation [65, 66, 1, 59].

4.11.1. Optimum Intra- and Interperiodic Processing

During separate processing, the received signal is divided into sections with a duration equal to the repetition period of modulation (for brevity we will call these sections periods of reflected signal), and each of these sections are processed as if it were unique, i.e., it is multiplied by the support signal and is integrated or is passed through a corresponding reducing filter. The character of the optimum processing of separate periods is determined by the results of Section 4.9, relating to a case of single transmission, since each separate period of modulation can be considered as a single transmission.

The effectiveness of a circuit of intraperiodic processing is determined by the output value of the signal-to-interference ratio, which we will designate by q and which is determined by formula (4.10.22). In paragraph 4.10.2 it was shown that with steep-drooping spectral densities of modulation, near to rectangular, the replacement of the optimum support signal by the expected signal does not lead to an essential decrease of the signal-to-interference ratio q. In connection with this, for modulation spectra satisfying the shown condition, in the presence of passive interferences the same processing can be used as in the presence of noises only.

As a result of intraperiodic processing with the use of support signal $Z(t,\tau)$ we obtain a sequence of magnitudes f_j $(j=1, \ldots, n)$

$$f_j = \int_{(j-1)T_r}^{jT_r} Z(t,\tau)\, y(t)\, dt, \qquad (4.11.1)$$

which must then be subjected to interperiodic processing. Magnitudes f_j are complex. The real φ_j and imaginary ψ_j parts of these magnitudes due to the assumed normality of the received signal are distributed by normal law. Joint distribution of magnitudes $\varphi_1, \ldots, \varphi_n; \psi_1, \ldots, \psi_n$ can be represented in the form of [9]

$$p(\varphi_1, \ldots, \varphi_n; \psi_1, \ldots \psi_n) = \frac{1}{\pi^n |R_{jk}|} \exp\left\{-\sum_{j,k=1}^{n} W_{jk} f_j f_k^*\right\}, \tag{4.11.2}$$

where W_{jk} is the element of matrix $\|W_{jk}\|$, inverse to correlated matrix $\|R_{jk}\|$, elements of which are determined by equality

$$R_{jk} = \overline{f_j f_k^*}. \tag{4.11.3}$$

If P_{c}, $P_{\text{ш}}$ and $P_{\text{п}}$ are powers of signal, noise, and interference in the output of a system of intraperiodic processing, then

$$R_{jk} = P_{\text{c}} \rho(j, k) e^{-i(j-k)\theta} + P_{\text{ш}} \delta_{jk} + P_{\text{п}} r(j, k), \tag{4.11.4}$$

where $\rho(j,k) = \rho(jT_r - kT_r)$ is the coefficient of interperiodic correlation of the fluctuation of the signal from the target;

$r(j, k)$ is the coefficient of interperiodic correlation of reflections from passive interference;

$\theta = \Delta\omega_{\text{д}} T_r$ is the Doppler shift of signal phase for a period.

Further, it is convenient to unite noise and reflection from passive interferences under the general designation of interference and to introduce the coefficient of correlation

$$r_{\text{э}}(j, k) = \frac{P_{\text{ш}} \delta_{jk} + P_{\text{п}} r(j, k)}{P_{\text{ш}} + P_{\text{п}}}.$$

in (4.11.4) to reject immaterial factor $P_{\text{ш}} + P_{\text{п}}$ and, while preserving designation R_{jk}, for the function being obtained, we present it in the form

$$R_{jk} = q \rho(j, k) e^{i(j-k)\theta} + r_{\text{э}}(j, k). \tag{4.11.4'}$$

It is necessary to note that, inasmuch as the relations received depend on the results of a synthesis of an optimum method of processing for a single

transmission, the assumption concerning the fact that fluctuation does not distort the law of modulation of the reflected signal, as before remains in force. Therefore, during continuous emission, we consider that $\rho(j, j+1)$ and $r(j, j+1)$ differs little from one. In the case of a pulse signal, it is sufficient to require smallness of random changes in amplitude and phase for the pulse duration. With this, the degree of correlation between neighboring pulses can be completely arbitrary. Considering pulse signals widely used in practice, we will not in the future restrict the limits of change of $r(j, k)$ and $\rho(j, k)$.

The synthesis of optimum interperiodic processing of signal and the calculation of detection characteristics is conducted in full conformity with the method described in Section 4.1, with the only difference being that integrals with respect to time, in the final formulas, are replaced by sums. The relation of verisimilitude in the considered case (4.11.2) can be written in the form

$$\Lambda(f_1, \ldots, f_n) = K_0 \exp\left\{ q \sum_{j,k=1}^{n} v(j, k) f_j f_k^* \right\}, \tag{4.11.5}$$

where K_0 is not dependent on f_j coefficient;

n is the number of jointly processed periods;

$v(j, k)$ is determined by equation

$$\sum_{l=1}^{n} v(j, l)\left[r_0(l, k) + q\rho(l, k) e^{i(l-k)\phi}\right] = \sum_{l=1}^{n} w(j, l) \rho(l, k) e^{i(l-k)\phi}, \tag{4.11.6}$$

being a discrete analog of equation (4.2.4).

In (4.11.6) $\|w(j,k)\|$ designates the matrix inverse to the correlated matrix $\|r_0(j, k)\|$. As can be seen from (4.11.5), for acceptance of a solution concerning presence of target, it is sufficient to compare with the threshold the magnitude

$$L = \sum_{j,k=1}^{n} v(j,k) f_j f_k^*. \tag{4.11.7}$$

The characteristic function of magnitude L is obtained by direct integration

with the use of distribution (4.11.2) and is determined by formula

$$\Psi(\eta)=\frac{1}{\left|\delta_{jk}-i\eta\sum_{l}v(j,l)R^{*}_{lk}\right|}. \qquad (4.11.8)$$

Applying to determinant (4.11.8) the method used in an analogous case in Section 4.1, we obtain

$$\Psi(\eta)=\exp\left\{\int_{0}^{i\eta}\sum_{j=1}^{n}G(j,j;\gamma)\,d\gamma\right\}, \qquad (4.11.8')$$

where $G(j,k;\gamma)$ is determined by equation

$$G(j,k;\gamma)-\gamma\sum_{l=1}^{n}G(j,l;\gamma)v_{1}(l,k)=v_{1}(j,k), \qquad (4.11.9)$$

and

$$v_{1}(j,k)=\sum_{l=1}^{n}v(j,l)R^{*}_{lk}. \qquad (4.11.10)$$

Inasmuch as when concluding (4.11.8') that no limitations were imposed on the form of function $v(j,k)$ this formula can be used for calculating the characteristics of detection of nonoptimal systems, the signal value in the output of which can be represented in the form of (4.11.7).

Let us turn now to detecting optimum methods of interperiodic processing for various specific cases. The solution of this problem, as one may see from (4.11.7) and (4.11.5), leads to the inversion of the correlated matrix of interference $\|r_0(j,k)\|$ and the solution of equation (4.11.6). This solution is fully analogous to the solution of the corresponding problems of section 4.9 and leads to fully analogous results.

We will start from a case of slow fluctuation of reflected signal, when $\rho(j,k)=1\ (1\leqslant j,k\leqslant n)$. As is easy to check, the solution of equation (4.11.5) in this case is written in the form [see (4.9.2)]

$$v(j,k)=\frac{1}{1+q_{1}}Z_{j}Z^{*}_{k}, \qquad (4.11.11)$$

where

$$Z_{j}=\sum_{l=1}^{n}w(j,l)\,e^{il\theta} \qquad (4.11.12)$$

$$q_1 = q\sum_{j=1}^{n} Z_j^* e^{ij\theta} = q\sum_{j,k=1}^{n} \omega(j,k)\, e^{i(j-k)\theta}. \qquad (4.11.13)$$

Substituting (4.11.11) in (4.11.7), we obtain

$$L = \frac{1}{1+q_1}\left|\sum_{j=1}^{n} Z_j f_j\right|^2, \qquad (4.11.14)$$

whence it is clear that optimum processing is reduced to formation of real and imaginary parts of the sum, in (4.11.14), and the summation of their squares. For a final explanation of the character of these operations it is necessary to reverse the correlated matrix of interference and to calculate the weight factors Z_j $(j=1, \ldots, n)$.

Elements of matrix $\|\omega(j,k)\|$ satisfy equation

$$\sum_{l=1}^{n} \omega(j,l)\, r_0(l,k) = \delta_{jk}. \qquad (4.11.15)$$

If $r_0(l, k)$ depends only on difference $l - k$ (the fluctuation of interference, as was already noted in Chapter 1, during solution of detection problems can be considered stationary), then, as is easy to see, $w(j, k)$ possesses the following properties of this equation:

$$\omega(j,k) = \omega(k,j) = \omega(n-j+1,\ n-k+1). \qquad (4.11.16)$$

Not imposing any limitations on the relationship between observation time $T = nT_r$ and correlation time of interference $\tau_{\text{кор}}$, we can solve equation (4.11.15) only for a particular case of fractional - rational spectral density $S_0(\lambda)$, corresponding to the function of correlation $r_0(j, k)$. Let us remember that the result of discrete Fourier transform of function $r(l)$ [67] is called the spectral density of a stationary random sequence with correlation function $r(l)$,

$$S(\lambda) = \sum_{l=-\infty}^{\infty} r(l)\, e^{-il\lambda}, \qquad (4.11.17)$$

to which corresponds the following inverse transformation:

$$r(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\lambda)\, e^{il\lambda}\, d\lambda. \qquad (4.11.18)$$

This inverse transformation frequently is conveniently recorded in the form of an integral with respect to single circle $|z|=1$ on complex plane $z=e^{i\lambda}$:

$$r(l)=\frac{1}{2\pi i}\oint\limits_{|z|=1} S'(z)\,z^{l-1}\,dz, \tag{4.11.18'}$$

where

$$S'(z)=S\left(\frac{1}{i}\ln z\right).$$

Thus, we will assume that spectral density $S_0(\lambda)$ has the form

$$S_0(\lambda)=\frac{1}{|P(e^{i\lambda})|^2}=\frac{1}{|a_0+a_1e^{i\lambda}+\ldots+a_me^{im\lambda}|^2} \tag{4.11.19}$$

and that the number of jointly processed periods $n>2m$. Substituting (4.11.19) in (4.11.18), and then in (4.11.15), we obtain

$$\frac{1}{2\pi i}\sum_{l=1}^{n} w(j,l)\oint\limits_{|z|=1}\frac{z^{l-k-1}}{|P(z)|^2}\,dz=\delta_{jk}. \tag{4.11.20}$$

We will apply to both parts the operator

$$\sum_{\nu,\mu=1}^{n} a_\nu a_\mu T(\nu,k)\,T(-\mu,k),$$

where $T(\nu,k)$ is the translation operator on ν of periods with respect to argument k (similar to the method used in paragraph 4.3.4). Inasmuch as, for the left part, such a transformation is reduced to multiplication of the integrand by $|P(z)|^2$, we obtain

$$w(j,k)=\sum_{\nu=0}^{m} a_\nu a_{\nu+k-j}. \tag{4.11.21}$$

This result is true only when $m+1\leqslant k\leqslant n-m$, since during other k, the application to the right side (4.11.20) of operator T(m, k) conducts this right side beyond the limits of interval (1, n), in which it is determined. Thus, left uncalculated are the left upper and right lower angles of the submatrix of the matrix $\|w(j,k)\|$ of the order of m. To calculate these submatrices we will use the following method. The assumption made that $n>2m$, allows us during $k\leqslant m+1$ to apply to (4.11.20) operator $\sum\limits_{\nu=0}^{m} a_\nu T(-\nu,k)$. With this, in (4.11.20) under the integral in

the denominator remains the function

$$P^{*}(z)=a_0^{*}+a_1^{*}\frac{1}{z}+a_2^{*}\frac{1}{z^2}+\ldots+a_m^{*}\frac{1}{z^m},$$

not having, if we accomplish the process physically, zeroes inside a single circle. The integrand in this case can have only a pole in zero of the order of $\nu=k-l-m-1$. Attributing various values to k starting with m + 1, we obtain for w(j, k) a system of equation of the form

$$\sum_{\nu=1}^{l} w(j,\nu)C_{l-\nu}=a_{m+l-j},\quad l=1,2,\ldots, \tag{4.11.22}$$

where C_ν is the deduction of the integrand at point z = 0.

System of equations (4.11.22) allows us consecutively, by columns, to calculate elements of matrix $\|w(j,k)\|$.

The transformations used above are useful also in the case where in the numerator in (4.11.19) there is also a polynomial. However, in this case they do not lead to final solution, but only somewhat simplify the system of equations.

Application of the method used of transformation of correlated matrices is limited also by condition $n>2m$. In a case of spectral densities of a low order this condition is usually fulfilled. When $n<2m$ apparently the only method of solving the problem is the well-known method based on Cramer's rule. Solution for n = 2, 3, 4 is found by this method in [9].

Using the described method of transformation of matrices, we will find optimum operations for a case when $S_0(\lambda)$ is approximated in the following manner:

$$S_0(\lambda)=\frac{A(\alpha)}{|e^{i\lambda}-\alpha|^{2m}}=\frac{A(\alpha)}{(1-2\alpha\cos\lambda+\alpha^2)^m}, \tag{4.11.23}$$

where $A(\alpha)$ is a coefficient, standardizing $r_0(l,l)$ to unity.

Spectral density (4.11.23) is a discrete analog of spectral density $\frac{A}{\left[1+\left(\frac{\omega}{2\Delta f}\right)^2\right]^m}$, convergent at $m\to\infty$ and fixed band to Gaussian. With various values of m and α function (4.11.23) can be used for approximation of a broad class of spectral densities of interference. The corresponding function of correlation has the form

$$r_0(j,k)=\alpha^{|j-k|} \text{ when } m=1,$$

$$r_0(j,k)=\alpha^{|j-k|}\left(1+|j-k|\frac{1-\alpha^2}{1+\alpha^2}\right) \text{ when } m=2.$$

If m is infinitely increased, preserving band $\Delta\lambda$ with respect to level 0.5 of spectral density $S_0(\lambda)$ of the constant, then

$$S_0(\lambda)\rightarrow\frac{e^{a\cos\lambda}}{I_0(a)};\ r_0(jk)\rightarrow\frac{I_{|j-k|}(a)}{I_0(a)}, \tag{4.11.24}$$

where

$$a=\frac{\ln 2}{1-\cos\Delta\lambda}.$$

For the considered approximation, with the help of (4.11.21) and (4.11.22) we obtain

$$w(j,k)=\frac{(-\alpha)^{|j-k|}}{A(\alpha)}\sum_{\nu=0}^{k-1}\binom{m}{\nu}\binom{m}{|j-k|+\nu}\alpha^{2\nu}. \tag{4.11.25}$$

In order to simplify recording, in (4.11.25) it is assumed that $\binom{m}{\nu}=0$ when $m<\nu$. Substituting (4.11.25) into (4.11.12), when $j<m$ we have

$$Z_j=\frac{(1-\alpha e^{i\theta})^m}{A(\alpha)}e^{ij\theta}\sum_{\nu=0}^{j-1}\binom{m}{\nu}(-\alpha e^{i\theta})^{\nu}. \tag{4.11.26}$$

When $m\leqslant j\leqslant n-m$ the sum in (4.11.26) is replaced by $(1-\alpha e^{-i\theta})^m$, and the values of Z_j when $j>n-m$ can be determined from (4.11.26) with the help of the property of symmetry

$$Z_{n-j+1}=e^{i(n+1)\theta}Z_j,$$

ensuing from (4.11.16).

Operations on the received signal, determined by relationships (4.11.14) and (4.11.26), in the general case can be carried out only with the help of complex computers directly carrying out all necessary mathematical transformations of magnitudes f_j. At the same time it is very desirable to consider the possibility of realizing these operations with the help of radiotechnical means, which contemporary radar has available. Such possibilities exist for certain specific cases.

We will assume that neighboring periods of interference are strongly correlated,

so that $a \approx 1$. With this, the output value of signal L for an optimum system can be represented in the form

$$L = \left| \sum_{j=m+1}^{n} e^{ij\theta} \sum_{\nu=1}^{m} \binom{m}{\nu} (-1)^{\nu} f_{j-\nu} \right|^2. \qquad (4.11.27)$$

The internal sum in this formula represents the difference of the m-th order of magnitudes $f_{j-m}, f_{j-m+1}, \ldots, f_j$. It follows from this that optimum operations (14.11.27) can be carried out with the help of m-multiple period-by-period subtraction with subsequent coherent summation of remainders and formation of the square of the modulus.

For signals from all distances m-multiple subtraction can be simultaneously realized, using m of the subtracting potentioscope of systems of period-by-period compensation (PPC) on ultrasonic delay lines [66].

A deficiency of the first variant is the presence of "background" of potentioscopes connected with the irregularity of the sensitive layer covering the target and leading to an additional increase in the remainders of subtraction, camouflaging the target.

In systems with delay lines, an additional increase of remainders occurs due to the instability of the delay. A deficiency of this kind of system is also the difficulty of changing the period applied in certain radar stations to combat blind speeds. A positive characteristic of systems with delay lines is the fact that in them subtraction can be carried out on an intermediate frequency and, even if subsequent accumulation is produced on an intermediate frequency, it is possible for the formation of the square of the modulus in (4.11.27) to use the square-law detector. With this, all the processing system is single-channel, while in the system with potentioscopes the presence of two channels for the formation of the square of the modulus is necessary.

In general, none of the compared PPC systems possesses decisive advantages; therefore, each of them is used depending upon the specific conditions of the work of a given radar. For the coherent accumulation of remainders of subtraction we

can also use a potentioscope with accumulation of charge (or delay line with feedback).

The general form of functional diagram of a receiving mechanism with low-frequency accumulators is shown in Fig. 4.24,a. The received signal proceeds to a filter of optimum intraperiodic processing (reducing filter) and then, with the help of mixers with sine and cosine reference voltage, is distributed between two quadrature channels. In each channel there occurs a suppression of interference with the help of a PPC system, after which there is carried out a compensation of Doppler shifts of phase of signal from the target and the formation of real and imaginary parts of certain components in (4.11.27), then accumulation and formation of the square of modulus. During the use of systems of interference suppression and accumulators working on an intermediate frequency, the diagram is considerably simplified and is reduced to the form shown in Fig. 4.24,b.

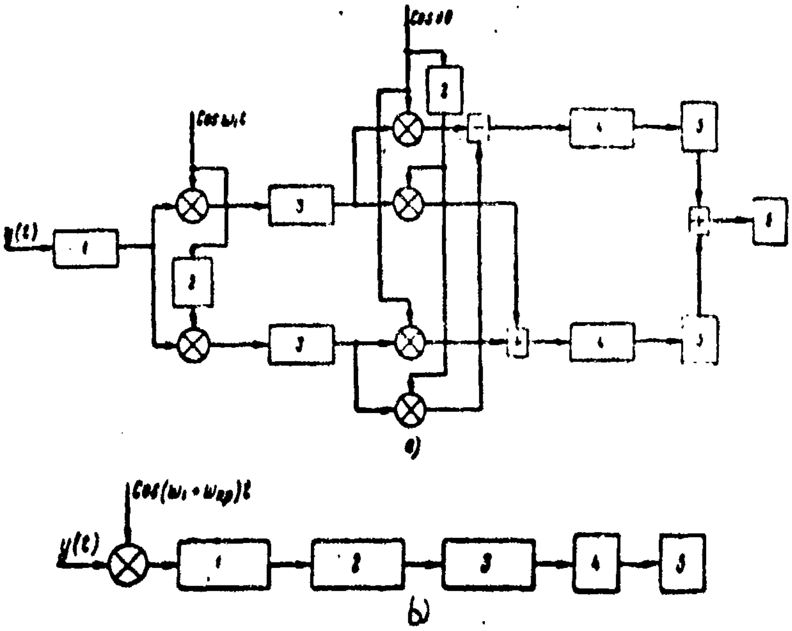

Fig. 4.24. Functional diagram of an optimum system of detection with interference suppression.
a) on low frequency: 1) diagram of intraperiodic processing (reducing filter); 2) phase inverter on 90°; 3) interference suppression device; 4) accumulator; 5) square-law generator; 6) relay.
b) on intermediate frequency: 1) system of intraperiodic processing; 2) interference suppression device; 3) accumulator; 4) square-law detector; 5) relay.

It is possible to carry out all operations without the use of a potentioscope and delay lines with the help of filters, pulse circuits of comparison, and similar elements; however, such a circuit will be able to process the signal only from one definite distance, as a result of which the advantages, on which we calculated, changing to the consideration of separate processing of periods, are completely lost.

It is necessary to note that the diagrams of Fig. 4.24 are calculated on fully defined values of phase shift, i.e., at a fully defined target speed. If the speed is unknown, the system of detection should consist of a combination of channels of the form of Fig. 4.24. Branching to channels can occur after suppression of interference. During the examination we did not also consider the Doppler shift of the interference, i.e., we consider passive interference of something motionless or moving with known speed, the knowledge of which allows us to compensate for the Doppler shift of the interference ("stop" of interference) before beginning interperiodic processing.

We considered in detail the case, most interesting for practice, of strongly correlated interference. In the other limiting case when $\alpha = 0$ optimum operations are turned into the usual coherent accumulation. During intermediate α, as was already noted, optimum operations for the considered case cannot lead to a form useful for technical realization with the help of known electronic circuits.

In the considered case of slow fluctuation of target, one can obtain a very graphic solution, if one were to assume that the angular dimensions of the interference exceed the width of the beam so that interference begins to be observed long in advance (at least, for several times the correlations of the interference) before the beginning of the reception of signal from target and continues to be observed after the reception of signal from target is completed. With this, it manages to be free from the influence of boundary conditions at the ends of the interval (0,T),

considerably complicating solution in a case when it was assumed that outside this interval, signals from interference and from the target are not observed. Considering $T \gg \tau_{\text{ип}}$, a solution of equation (4.11.15) can be obtained by Fourier transform:

$$w(j,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{S_0(\lambda)} e^{i\lambda(j-k)} d\lambda. \qquad (4.11.28)$$

Designating the value of the bunch envelope g(t) at moments $t_k = kT_r$ by $g_{k-\nu}$ (ν is the moment of passage of the front edge of the diagram through target), taking into account (4.11.28) we have

$$Z_j = \sum_{-\infty}^{\infty} w(j,l) e^{il\theta} g_{k-\nu} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{G(\lambda-\theta)}{S_0(\lambda)} e^{i(j-\nu)\lambda + i\nu\theta} d\lambda, \qquad (4.11.29)$$

where $G(\lambda)$ is the spectrum of sequence g_l.

Substituting (4.11.29) in (4.11.14), we obtain

$$L \sim \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} G^*(\lambda-\theta) \frac{F(\lambda)}{S_0(\lambda)} e^{ij\lambda} d\lambda \right|^2. \qquad (4.11.30)$$

where $F(\lambda)$ is the spectrum of sequence f_j.

Optimum operations, corresponding to (4.11.30), can be carried out in the functional diagram represented in Fig. 4.24,a. The system of interference suppression in the given case is a rejector filter with frequency response $\frac{1}{S_0(\lambda)}$, and the accumulator is replaced by pulse filter $G^*(\lambda)$, coordinated with the form of the bunch. Comparison of signal at input of relay with the threshold occurs in this case continuously. Exceeding threshold at any moment indicates the presence of a target in a corresponding direction.

Frequency response $\frac{1}{S_0(\lambda)}$ is not realizable since the corresponding phase response is equal identically to zero. However, accurate realization of this characteristic in practice is not required. Calculation of detection characteristics (4.11.2) shows that a main role is played by the behavior of frequency response in small environment θ. Regarding phase response, it can always be constructed, by a simple delay corresponding to environment θ which can be considered when determining direction to the revealed target.

If the spectral density of the interference is approximated by function (4.11.23), then

$$\frac{1}{S_0(\lambda)} = |1 - \alpha e^{-i\lambda}|^{2m}$$

coincides with the modulus of frequency response of a system of 2m-multiple period-by-period subtraction, in which there is introduced an additional weakening, α times, of the delayed-in-period signal. Such weakening can be very simply carried out in a system with an ultrasonic delay line. In a system of subtraction with potentioscope, some kind of method to accelerate runoff of charge from target is required for this. Phase response of such a system of subtraction is expressed by formula

$$\varphi(\lambda) = \arctan\frac{\alpha\sin\lambda}{1-\alpha\cos\lambda}$$

and has the form shown in Fig. 4.25.

As can be seen from the figure, in regions λ near π phase response during all α is near linear, where by the measure of increase in α the region of linearity expands. Most interesting in practice is the case when θ is near π (optimum speed of target), where the greatest suppression of interference is ensured. Precisely in these cases, as a system of interference suppression, can a system of period-by-period compensation of corresponding multiplicity with weakening of the delayed signal be used.

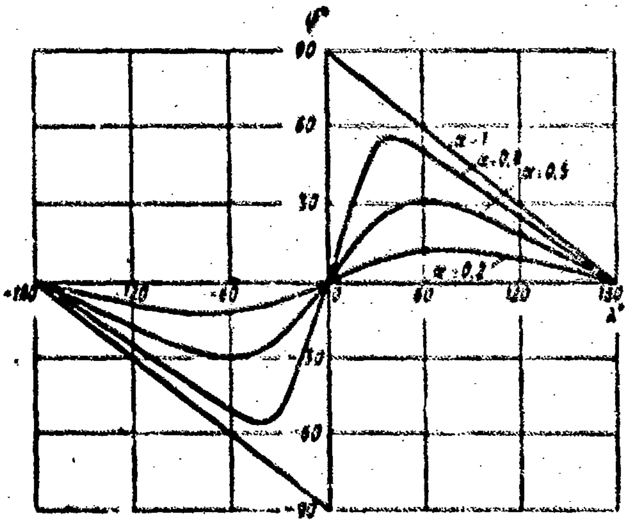

Fig. 4.25. Phase-frequency response of the subtracting mechanism with a feedback factor different than one.

It is to be noted that for the same spectral density (4.11.23) multiplicity of period-by-period subtraction is obtained in the given case twice as much as with limited observation time of the interference. This result is due to the influence of boundaries of the observation range. Conditions of the application of these two solutions were discussed above.

In connection with the use, for approximation, of spectral density $S_0(\lambda)$ of function (4.11.23) the following should be noted.

During α, near unity, when $S_0(\pi) = \frac{A(\alpha)}{(1+\alpha)^{2m}}$ is much less than the equivalent spectral density of noise $N_1 = \frac{P_ш}{P_ш + P_п}$, the use of such an approximation implies a disregard of noise as compared with interference. If $T \gg \tau_{кп}$, then the quality of detection is affected mainly by the behavior of function $S_0(\lambda)$ in small environment θ and, if θ is not too near the edges of the interval, the indicated disregard is fully permissible. In the case of narrow-band interference and a small observation time, disregard of noise can lead to incorrect conclusions about the character of optimum processing.

Let us consider now the case of fast fluctuations reflected from the target and from the interference of the signal. With this, both equation (4.11.6) and (44.11.15) can be solved with the help of Fourier transform. As a result of substitution of the solution into (4.11.7), we obtain

$$L = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_0(i\lambda)|^2 |F(\lambda)|^2 d\lambda = \sum_{j=1}^{n} |F_j|^2, \qquad (4.11.31)$$

where $F(\lambda)$ is the spectrum of sequence f_j $(j = 1, \ldots, n)$;

$$|H_0(i\lambda)|^2 = \frac{S(\lambda-\theta)}{S_0(\lambda)[S_0(\lambda) + qS(\lambda-\theta)]} \qquad (4.11.32)$$

-- is the spectrum of function v(j,k);

$S(\lambda)$ -- is spectral density corresponding to the function of correlation of fluctuation of signal

By F_j in (4.11.31) is designated the result of the transmission of sequence f_j through the pulse filter with frequency response $H_0(i\lambda)$. In accordance with (4.11.32)

filtration can be broken down into two stages: suppression of interference in the filter

$$|H_p(i\lambda)|^2 = \frac{1}{S_0(\lambda)},$$

turning interference into noise with uniform spectral density, and singling out signal with the help of filter

$$|H_n(i\lambda)|^2 = \frac{\frac{S(\lambda-\theta)}{S_0(\lambda)}}{1+q\frac{S(\lambda-\theta)}{S_0(\lambda)}}.$$

Optimum processing determined by formula (4.11.31) can be carried out with the help of a somewhat augmented functional diagram of Fig. 4.24,b, in which elements are used of interference suppression and signal accumulation, which work on an intermediate frequency. Changes in the diagram lead to the introduction between the detector and the relay of an incoherent accumulator in n periods, as which can be used, for example, a storing potentioscope.

If one were to attempt to use, during interperiodic processing, only mechanisms working on low frequency (type of potentioscope), then it is necessary to represent optimum operations (4.11.31) in the form of transformations above $\operatorname{Re} f_j$ and $\operatorname{Im} f_j$. To these transformations can be added a comparatively convenient form for technical embodiment only by means of using certain additional approximations.

If the signal-to-interference ratio is $q \ll 1$, so that $qS(\lambda-\theta) \ll S_0(\lambda)$ during all λ, then

$$|H_0(i\lambda)|^2 \approx \frac{S(\lambda-\theta)}{S_0^2(\lambda)}. \qquad (4.11.33)$$

In this case, optimum processing can be carried out with the help of the diagram in Fig. 4.24,a, augmented by an incoherent accumulator in which the mechanism of interference suppression must possess frequency response $1/S_0(\lambda)$ [cf. (4.11.30)] and the storing filter must have square of modulus of the frequency response $S(\lambda)$.

If $q \gg 1$, so that $qS(\lambda-\theta) \gg S_0(\lambda)$ during all λ, then

$$|H_0(i\lambda)|^2 \approx \frac{1}{S_0(\lambda)} \qquad (4.11.34)$$

and optimum processing leads to a summation of squares of the output of quadrature channels, in each of which is a rejector filter (4.11.34). This diagram is nearest to that used in practice, in which usually after period-by-period subtraction of the corresponding multiplicity incoherent accumulation immediately occurs. In connection with this one should note that condition $q \gg 1$, with which existing systems are near optimum, usually is not fulfilled.

Optimum processing leads to filtration in the two quadrature channels also and in a case if θ is a multiple of π (blind or optimum speed), and also (during the introduction of shift with respect to frequency on $\frac{\theta}{T_r}$ during separation to quadrature channels) if the interference is sufficiently broad-band, so that it is possible to consider $S_0(\lambda) \approx S_0(\theta)$ in regions where $S(\lambda - \theta)$ is noticeably different than zero. Taking into account shift to θ in this case

$$|H_0(i\lambda)|^2 \approx \frac{S(\lambda)}{1 + \frac{q}{S_0(\theta)} S(\lambda)}. \qquad (4.11.35)$$

The filter of interference suppression, as one may see from (4.11.35), in this case disappears, and the characteristic of the storing filter is turned into a discrete analog of characteristic (4.3.8), which is completely understandable since the approximation used signifies, actually, the replacement of interference by equivalent white noise.

We have considered optimum methods of interperiodic processing of the signal in the presence of passive interferences, various simplified approaches to these methods of processing, and touched upon the question of possible means of technical realization of these methods. The next stage in the investigation is the obtaining of equations of detection characteristics, corresponding to optimum processing, so that it is possible, by means of a comparison of optimum and existing systems, to reach conclusions concerning the expediency of a transition to optimum methods of processing.

4.11.2. Characteristics of Detection During Optimum Interperiodic Processing

The calculation of the characteristics interesting us is conducted with the help of relationships (4.11.8') -- (4.11.10), where the approaches used, a group of considered specific cases and methods of solution, are completely analogous with those used when examining analogous problems in paragraphs 4.4.1 and 4.10.1. In connection with this, we will reduce, partially, the explanations pertaining to methods of solution of equation (4.11.9) and, in detail, will stop only on interpretation of results.

During the slow fluctuation of the reflected signal, the probabilities of correct detection and false alarm are connected with the relationship

$$D=F^{\frac{1}{1+q_1}},$$

where q_1 is determined by formula (4.11.13) or during calculation of the form of the bunch in accordance with (4.11.30) by the following expression

$$q_1=q\,\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{|G(\lambda-\theta)|^2}{S_0(\lambda)}\,d\lambda. \qquad (4.11.36)$$

If the character of processing differs from optimum, so that Z_j in (4.11.14) no longer is determined by formula (4.11.12), and $\frac{1}{S_0(\lambda)}$ and $G(\lambda-\theta)$ in (4.11.30) are replaced by some frequency responses $H_p(i\lambda)$ and $H_n(i\lambda)$, formulas (4.11.13) and (4.11.36) are replaced, respectively, by the following:

$$q_1=q\,\frac{\left|\sum\limits_{j=1}^{n}Z_je^{-ij\theta}\right|^2}{\sum\limits_{j,k=1}^{n}Z_jZ_k^*r_0(j,k)}=q\,\frac{|H_1(i\theta)|^2}{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}S_0(\lambda)|H_1(i\lambda)|^2d\lambda}, \qquad (4.11.37)$$

$$q_1=q\,\frac{\left|\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}H_p(i\lambda)H_n(i\lambda-i\theta)G(\lambda-\theta)d\lambda\right|^2}{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|H_p(i\lambda)H_n(i\lambda-i\theta)|^2S_0(\lambda)\,d\lambda}, \qquad (4.11.38)$$

In (4.11.37) $H_1(i\lambda)$ designates the sum $\sum\limits_{j=1}^{n}Z_je^{-ij\theta}$, which it is possible to consider as frequency response of some filter. These formulas can be used to appraise the

quality of systems of detection differing from optimum.

If spectrum $G(\lambda)$ is narrow as compared with the spectrum of interference (duration of bunch $T \gg \tau_{\text{кп}}$) and with frequency response of filter, then from (4.11.38) we obtain

$$q_1 \approx q \frac{\left|\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} H_{\text{н}}(i\lambda) G(\lambda)\, d\lambda\right|^2}{S_0(\theta)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |H_{\text{н}}(i\lambda)|^2\, d\lambda}, \qquad (4.11.39)$$

from which it is clear that the signal-to-interference ratio does not depend in this case on the form of frequency response $H_{\text{р}}(i\lambda)$ of the rejector filter suppressing interference. From the frequency response of the storing filter in practice it is sufficient to require matching with spectrum $G(\lambda)$ with respect to the band. If $H_{\text{н}}(i\lambda) \approx G^*(\lambda)$, then

$$q_1 \approx q \frac{1}{S_0(\theta)} \cdot \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |G(\lambda)|^2\, d\lambda = q \frac{n_{\text{эф}}}{S_0(\theta)}, \qquad (4.11.40)$$

where $n_{\text{эф}}$ is the effective number of pulses in the bunch.

Approximate formula (4.11.40) is very graphic and can be used with success during practical calculations.

The comparative effectiveness of various methods of coherent interperiodic processing essentially depends on the form of spectral density of interference and on the relationship between the power of the interference and the noise in the output of a system of intraperiodic processing. As an example, illustrating this position, let us consider the effectiveness of certain methods of processing with exponential and Gaussian functions of correlation of a signal reflected from passive interferences.

In Fig. 4.26,a is presented the dependence of ratio $\frac{q_1}{q}$ for three forms of processing (optimum, coherent summation, and subtraction of maximum multiplicity n — 1) on $\Delta f_{\text{д}} T$ when $\theta = (2k+1)\pi$ and n = 16. All three methods give very close results. When $\Delta f_{\text{д}} T \ll 1$ coherent summation gives the same results as optimum processing. This agrees with the above-noted disappearance of dependence of q_1 on $H_{\text{р}}(i\lambda)$, occurring

when $\Delta f_{\text{п}} T \gg 1$. Period-by-period subtraction of maximum multiplicity for the considered approximation of the function of correlation differs considerably from optimum processing and during all $\Delta f_{\text{п}} T$ gives approximately a double loss.

Fig. 4.26,b presents an analogous relation for the Gaussian function of correlation, constructed for coherent accumulation and subtraction of maximum multiplicity (which in this case is near optimum processing during absence of noises) with various values of the interference-to-noise ratio $q_{\text{п}}$. With large $\Delta f_{\text{п}} T$ coherent summation in this case gives an advantage as compared with subtraction. However, as the spectrum of the fluctuations of the interference narrows during large $q_{\text{п}}$ we begin to see the advantages of multiple subtraction which when $q_{\text{п}} = 1\,000$, for example, ensures maximum gain to 11 db at $\Delta f_{\text{п}} T \approx 5$. Then this gain starts to diminish, and, finally, again advantage turns out to be on the side of coherent summation. This is explained by the fact that with strong correlation of interference, for its removal almost any operation can be successfully used, and the suppression of noises starts to play a main role, best ensured during coherent summation.

We cannot consider similar dependences in the total view, while not specifying spectral density of fluctuations and not assuming a broad-band nature of the interference. In connection with this, the study of fluctuation characteristics of signals from various kinds of passive interferences, the detection of the most characteristic, reliable approximations for these characteristics, and the numerical calculation of parameters of corresponding optimum systems and characteristics of detection is of great value.

During the fast fluctuation of reflected signal, a solution of equation (4.11.9) can be obtained by Fourier transform. As a result of the substitution of solution into (4.11.8') for the characteristic function of random variable L, we obtain

$$\Psi(\eta) = \exp\left\{-\frac{n}{2\pi}\int_{-\infty}^{\infty} \ln\left[1 - i\eta\,|H_0(i\lambda)|^2 S_{ss}(\lambda)\right] d\lambda\right\}. \qquad (4.11.41)$$

where $S_{\text{вх}}(\lambda)$ is the spectral density of sequence f_j, proceeding to input of the detection system.

In the presence of target $S_{\text{вх}}(\lambda) = qS(\lambda-\theta) + S_0(\lambda)$, during the absence of target $S_{\text{вх}}(\lambda) = S_0(\lambda)$. In its form, formula (4.11.41) coincides with formula (4.4.7), and for characteristics of detection, corresponding to the found characteristic function analogous approximation can be fully used. Semi-invariants of the distributive law corresponding to this characteristic function are determined by formula

$$\varkappa_\nu = (\nu-1)!\,\frac{n}{2\pi}\int_{-\pi}^{\pi}[|H_0(i\lambda)|^2 S_{\text{вх}}(\lambda)]^\nu d\lambda = n\varkappa'_\nu . \qquad (4.11.42)$$

Finding, with the help of (4.11.42), skewness and excess coefficients of the considered law, it is simple to see that with growth n these coefficients approach zero and, consequently, distributive law converges to normal. With this, for calculating detection characteristics, we can use normal approximation (4.4.10) or, for more accurate calculations, Edgeworth's series (4.4.11). It is necessary to note that in the derivation of formula (4.11.41) no assumptions were made about the form of frequency response $H_0(i\lambda)$, for which this formula, and also all ensuing from it can be used for calculating the characteristics of nonoptimal systems of detection and, in particular, the systems of period-by-period compensation considered below.

Let us consider certain comparatively simple approximations for the characteristics of an optimum detection system ensuing from (4.11.41). If nT_r is great as compared with the time of correlation of signal and interference, then, using (4.4.10), a very simple expression can be obtained for the threshold signal-to-interference ratio for one period when D = 0.5. With this, $\Phi^{-1}(D) = 0$, and, if we assume additionally that for $H_0(i\lambda)$ is used approximation (4.11.33), from (4.4.10) we obtain

$$q = \frac{\Phi^{-1}(1-F)}{\sqrt{\frac{n}{2\pi}\int_{-\pi}^{\pi}\frac{S^2(\lambda-\theta)}{S_0^2(\lambda)}d\lambda}}. \qquad (4.11.43)$$

In the case of interference, broad-band as compared with fluctuations of

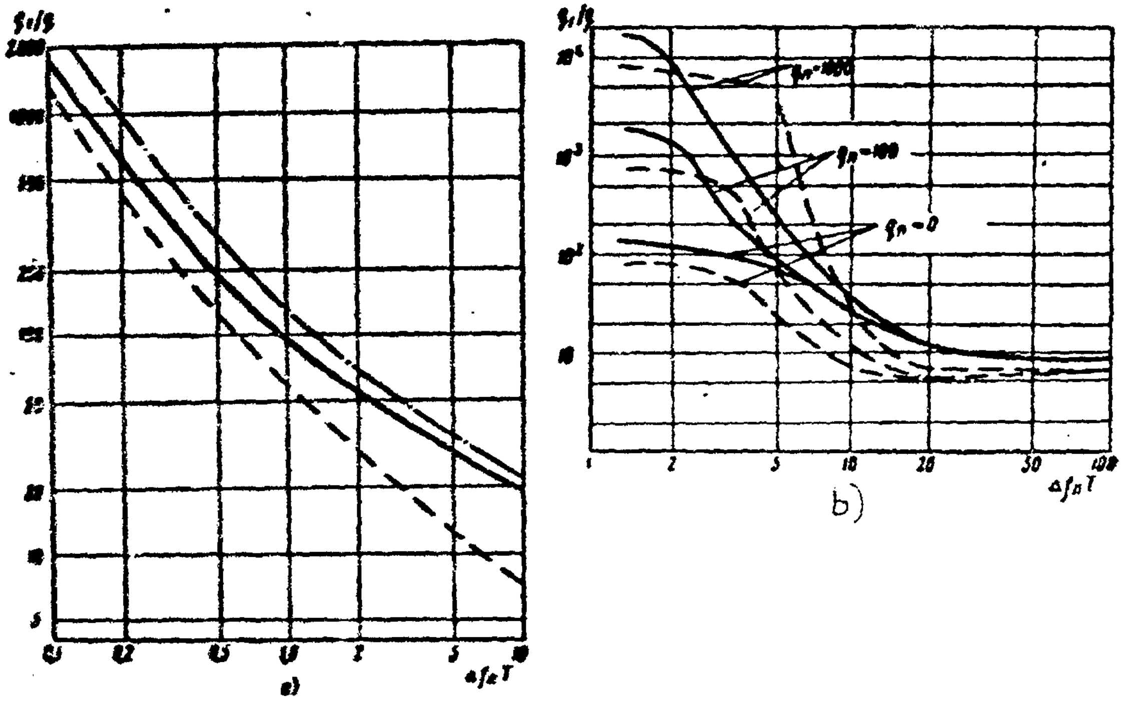

Fig. 4.26. Dependence of $\frac{q_1}{q}$ on $\Delta f_{\text{л}} T$: a) exponential function of correlation of interference; b) Gaussian function of correlation of interference: -.-.- optimum processing; - - - - subtraction of maximum multiplicity; ——— coherent summation.

signal

$$q \approx \Phi^{-1}(1-F)\frac{S_0(0)}{n}\sqrt{\Delta f_1 T}, \qquad (4.11.43')$$

where T is time of observation, and band Δf_1 is determined by equality

$$\Delta f_1 = \left[\frac{T_r}{2\pi}\int_{-\infty}^{\infty} S^2(\lambda)\,d\lambda\right]^{-1}.$$

With a small effective width of fluctuation spectrum $(\Delta f_c T_r \ll 1)$ band Δf_1 coincides with accuracy up to a constant factor determined by the form of fluctuation spectrum with Δf_c (for example, for rectangular spectrum $\Delta f_1 \approx \Delta f_c$; for exponential function of correlation of fluctuation $\Delta f_1 \approx 2\Delta f_c$). During large $\Delta f_c T_r$ band $\Delta f_1 \to \frac{1}{T_r}$. It is interesting to note that in formula (4.11.43'), as also in (4.11.40), was the ratio of spectral density of interference at point $\lambda = 0$ to the number of jointly processed periods of signal.

Usually the signal-to-interference ratio q for the period turns out to be very small, since the reflecting surface of passive interferences, occurring in a resolution range with respect to distance, in practice can many times exceed the reflecting surface of the target. Therefore, of greatest interest is the case when threshold q is sufficiently small. When $q \ll 1$ in (4.4.10), it is possible to disregard the difference of dispersions $\varkappa_3'$ and $\varkappa_{3n}'$ and to receive for q an approximate formula analogous to (4.11.43), but valid during arbitrary D;

$$q \approx \frac{\Phi^{-1}(1-F)+\Phi^{-1}(D)}{\sqrt{\frac{n}{2\pi}\int_{-\infty}^{\infty}\frac{S^2(\lambda-0)}{S_0^2(\lambda)}\,d\lambda}} \approx [\Phi^{-1}(1-F)+\Phi^{-1}(D)]\times \qquad (4.11.44)$$

$$\times\frac{S_0(0)}{n}\sqrt{\Delta f_1 T}.$$

The last equality in this formula is valid during broad-band interference.

For a case of broad-band interference, an approximation valid even during comparatively small n, can be obtained by approximating spectral density of the fluctuations of useful signal $S(\lambda)$ by a П-shaped curve having a width of $2\pi\Delta f_c T_r$. With this, making the same transformation as during the derivation of (4.4.13), we

obtain

$$q \approx \Delta f_c T_r S_0(\theta)\left(\frac{K^{-1}_{2\Delta f_c T}(F)}{K^{-1}_{2\Delta f_c T}(D)} - 1\right). \qquad (4.11.45)$$

The formulas obtained allow us to trace the dependence of the threshold signal-to-interference ratio q on the relationship between time of observation T and width of the fluctuation spectra of interference and signal. Let us consider such dependence for a particular case of exponential functions of correlation of these fluctuations, using, when $\Delta f_c T \gg 1$ and $\Delta f_{\text{п}} T \gg 1$ a normal approximation, and producing interpolation in the interval between $\Delta f T \approx 0$ and $\Delta f T \gg 1$, where formulas received for boundary cases of fast and slow fluctuations are not valid. Even during the use of such approximation, a sufficiently accurate calculation of this dependence for optimum processing is connected with great calculating difficulties and requires, in particular, the numerical solution of equations of a high order relative to q. In connection with this, we will omit here all intermediate calculations and we will turn directly to a discussion of the results of the calculation presented graphically in Fig. 4.27, when D = 0.9, $F = 10^{-4}$.

Dependence $q\ (\Delta f_c T)$ when $\theta = (2k+1)\pi$ in many respects is analogous to dependence $q_0(\Delta f_c T)$ of Fig. 4.7 and, in particular, also has a minimum at some value of $\Delta f_c T$, decreasing by measure of decrease of $\Delta f_{\text{п}} T_r$ (strengthening of correlation of interference). With this, threshold q diminshes.

The dependence of q on $\Delta f_{\text{п}} T_r$ when $\theta = 2k\pi$ has another character. When $\Delta f_c T \gg 1$ magnitude q diminishes with decrease of $\Delta f_{\text{п}} T_r$, as also in the case of optimum speed; however during decrease of $\Delta f_c T$, when the signal becomes correlated more strongly than the interference, q grows during a decrease of $\Delta f_{\text{п}} T_r$. This is easily explained physically. During large $\Delta f_c T$ the signal has a wider spectrum than the interference, and, in spite of strong overlapping of these spectra during $\theta = 2k\pi$, after suppression of the interference a significant part of the energy of the signal remains unsuppressed, the greater the part, the narrower the spectrum of interference.

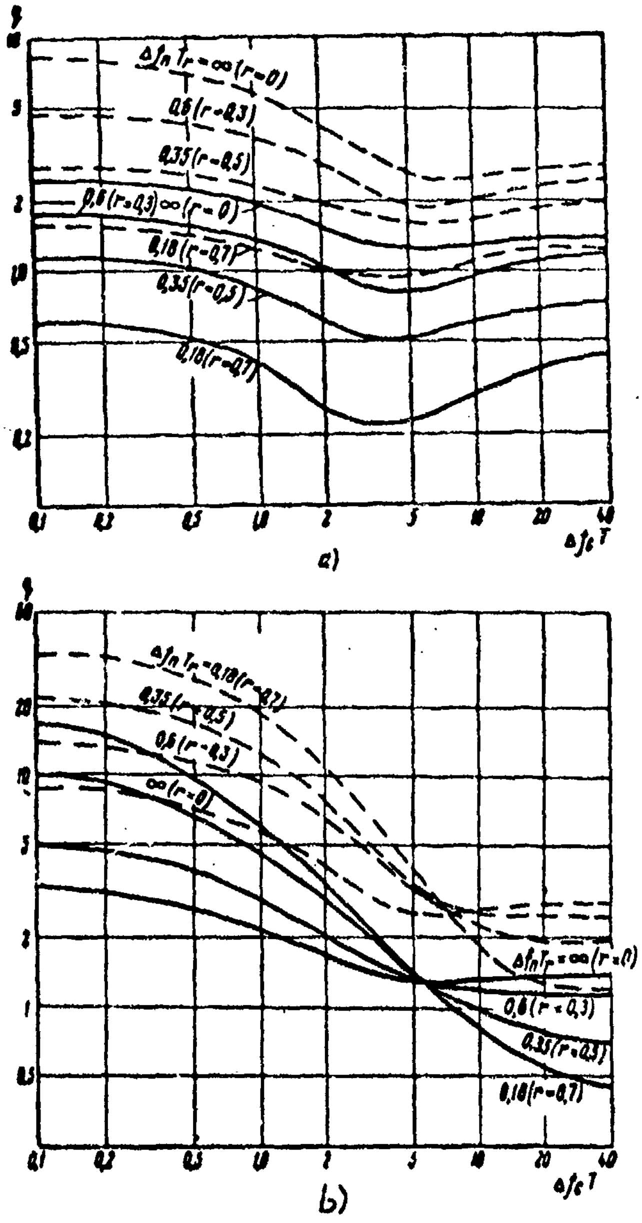

Fig. 4.27. Dependence of threshold signal-to-interference ratio q on width of fluctuation spectrum $\Delta f_c T$ for various $\Delta f_n T_r$: a) optimum speed of target; b) blind speed of target.

When the spectrum of the signal is narrower than the spectrum of the interference, its suppression leads to just as much or even stronger suppression of signal.

It is interesting to note that the best results during blind speed are obtained during a complete absence of correlation in neighboring periods, interference, and useful signal (we do not mention a case of correlated interference and uncorrelated signal, which, apparently, is unreal). This circumstance can be used in those cases when the speed of target relative to passive interferences is so small that the Doppler frequency turns out to be less than the width of the spectrum of the fluctuations of interference, and a selection of target based on speed is impossible. Such cases can take place during detection of slowly moving targets on a background of the earth's surface or artificial interferences, when the spectrum of interference is very wide due to the fast motion of the radar (for example, fixed on an aircraft).

To remove correlation of interference periods, we can retune the frequency from pulse to pulse to a magnitude somewhat larger than the width of the spectrum of modulation. If the correlation of useful signal is partially preserved then, as can be seen from the figure, losses in the magnitude of the threshold of q will constitute approximately 3 db. As follows from the results of Section 1.3, a resolving power comparable with the dimensions of the target, weakening in the correlation of the useful signal and interference will occur simultaneously with an increase in the difference of utilized frequencies.

We will estimate the possibility of such a method of increasing noise-resistance. With independent periods, optimum interperiodic processing consists of the summation of the squares of their envelopes. The equation of detection characteristics in this case is determined by formula (4.6.3), which during small q and $m \gg 1$ can be written in the form

$$q \approx \frac{\Phi^{-1}(1-F) + \Phi^{-1}(D)}{\sqrt{m}},$$

where m is the number of periods.

If D = 0.9 and F = 10^{-4}, then for reliable work when the interference exceeds the signal by 10 times, m = 2,500 periods will be required. In order to ensure the independence of all these periods due to retuning of frequency, it is necessary with a 10 milligram width of modulation spectrum to have a retuning range of 25,000 milligrams, which clearly is unreal. Actually it is sufficient to use periodic retuning with a period equal in order of magnitude to the time of correlation of the reflected signal. If we even consider the retuning range equal to 1,000 milligrams, then detection time will constitute 25 times the correlation, i.e., on the order of 3 sec for a target with a fluctuation spectrum width of 10 cps. The considered example shows that the possibilities of increasing noise-resistance in reference to passive interferences by means of decorrelating periods are very limited.

In conclusion of this division we will touch upon the question of the comparison of optimum separate processing of periods with the completely optimum processing considered in Sections 4.9 and 4.10. Unfortunately, there are no exhaustive results on this question at present. The only case which can be considered is the case of broad-band (as compared with useful signal) interference, when interference may be, in essence replaced by equivalent white noise. With this, according to (4.11.39), the form of the characteristic of the filter suppressing interference becomes immaterial and optimum interperiodic processing can be replaced by coherent accumulation. Also completely optimum processing in this case leads to multiplication by the periodic support signal and subsequent accumulation (see paragraph 4.9.3). The only difference consists in the form of support signal: in the case of completely optimum processing the form of signal takes into account the possibility of frequency selection and depends, in accordance with (4.10.29), on

$$\frac{e_0(\tau)}{2N_0}\sum_{-\infty}^{\infty} S_{\psi_0}(\tau,\ \Delta\omega_д - l\Omega_r) \approx \frac{e_0(\tau)T_r}{2N_0} S_u(\omega); \qquad (4.11.46)$$

and in the case of separate processing, the form of support signal, calculated only on one period of modulation, depends only on $S_{ue}(\omega)$. (In the derivation of (4.11.46)

was used a connection between spectral densities of a stationary random process and a sequence of values of this process at equidistant moments of time [67]).

As one should have expected, in this case the results of comparison depend on the form of function $S_{\text{м0}}(\omega)$. For a signal with rectangular spectral density of modulation, both forms of processing are completely equivalent. For a signal with slowly drooping spectral density

$$S_{\text{м0}}(\omega)=\frac{1}{\Delta f_{\text{м}}}\frac{1}{1+\left(\frac{\omega}{2\Delta f_{\text{м}}}\right)^2},$$

using (4.11.40), (4.10.29), (4.11.46), (4.10.22), and (4.9.27), we obtain an expression for a relative increase of signal-to-interference ratio q_1, due to optimum processing in the form

$$\Gamma=\frac{q_{1\text{опт}}}{q_{1\text{разд}}}\approx\frac{1+\frac{\gamma}{2}[1+S_{\text{п}}(\theta)]}{\sqrt{(1+\gamma)[1+\gamma S_{\text{п}}(\theta)]}},$$

where

$$\gamma=\frac{e_0(\tau)T_r}{2N_0\Delta f_{\text{м}}}.$$

Dependence $\Gamma(\gamma)$ with various values of spectral density of the fluctuations of interfering reflections $S_{\text{п}}(\theta)$ is shown in Fig. 4.28. When $\gamma\to\infty$

$$\Gamma\to\frac{1}{2}\left(\frac{1}{\sqrt{S_{\text{п}}(\theta)}}+\sqrt{S_{\text{п}}(\theta)}\right).$$

For the considered form of function $S_{\text{м0}}(\omega)$ optimum processing gives a substantial gain when $S_{\text{п}}(\theta)\ll 1$ and $S_{\text{п}}(\theta)\gg 1$. With increase in steepness of the droop of spectrum $S_{\text{м0}}(\omega)$ the magnitude of the gain decreases.

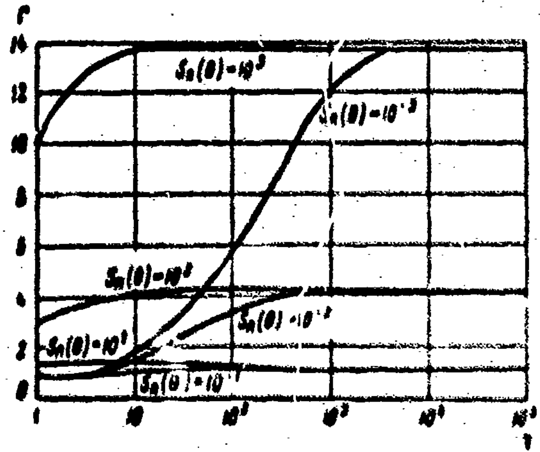

Fig. 4.28. Loss due to separate processing of periods.

As yet, we have not managed to consider a similar dependence in the case (important for a whole number of applications) of narrow-band interference.

4.11.3. Effectiveness of a System of Period-by-Period Compensation (PPC) with Internal Coherence

The most widespread method of protection from passive interference is period-by-period subtraction of any multiplicity with subsequent incoherent summation [66]. Systems of period-by-period compensation with internal and external coherence differ. In the system with internal coherence the signal after phase detection (or after frequency conversion, the subtracting mechanism, works on an intermediate frequency) proceeds to the system of period-by-period subtraction, in which the signal from motionless interference is compensated.

If the interference moves (for example, due to wind), then the system must include a mechanism to compensate for the speed of interference, ensuring a corresponding alignment of heterodyne. This circumstance considerably hampers use of a system with internal coherence, when the speed of interference is unknown, for example, during movement of the radar, whose speed we usually cannot measure accurately enough. In such cases it is possible, in general, to apply a system of compensation with external coherence [66], in which pulsations of signals are used from the target and from interference due to the difference of their speeds, and to input of the PPC system the signal moves from the output f the amplitude detector.

To the analysis of the effectiveness of PPC systems with internal coherence are devoted a number of works; however, usually as a standard of effectiveness the degree of suppression of interference in the subtracting mechanism has been used, but the characteristics of detection have not been considered. In [57] dependence $D(F)$ was calculated for a system of single subtraction having internal coherence without accumulation. Here let us consider the effectiveness of the PPC system with arbitrary multiplicity of subtraction m when $n \gg m$ (n is the number of jointly processed periods). The essential limitation, used during the analysis, is the requirement for

shortness of time of correlation of interference as compared with the time of observation.

Let us consider at first a system of period-by-period compensation with internal coherence. With this, we will assume that compensation is carried out in two quadrature channels or on an intermediate frequency. Single-channel compensation leads, as noted in [57], to essential loss, since with this, is lost one of the independent quadrature components of the signal and the relative magnitude of fluctuation is increased. In connection with this circumstance, two-channel compensation has found application in a whole number of stations using the PPC system on low frequency (potentioscope). A comparison of a single-channel and a two-channel system will be conducted the end of this division.

A system of m-multiple period-by-period compensation is a pulse linear filter with frequency response

$$H(i\lambda)=(1-e^{-i\lambda})^{m}. \qquad (4.11.47)$$

Designating by h_k the pulse reaction of this filter and considering that after subtraction there will be formed a sum of squares of the output voltages of quadrature channels and incoherent accumulation occurs, it is possible to represent the signal at input of relay in a form, analogous to (4.4.1):

$$L=\sum_{l=1}^{n}\left|\sum_{k=0}^{l}h_{l-k}f_k\right|^2.$$

Using the assumption that $m \ll n$, and producing the same conversion as during transition from (4.4.1) to (4.4.1'), we obtain

$$L=\sum_{j,\,k=1}^{n}v_{jk}f_jf_k^{*}, \qquad (4.11.48)$$

where

$$v_{jk}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(i\lambda)|^2e^{i\lambda(j-k)}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{\infty}|1-e^{-i\lambda}|^{2m}e^{i\lambda(j-k)}d\lambda. \qquad (4.11.49)$$

Presentation (4.11.48) coincides in form with (4.11.6); therefore, for obtaining detection characteristics relationships (4.11.8') — (4.11.10) can be used.

With fast fluctuation of the reflected signal, to calculate detection characteristics expression (4.11.41) can be used, which is correct, as already noted, during arbitrary frequency response of filter $H_0(i\lambda)$. Accordingly approximations for detection characteristics ensuing from this formula can be used. During small q (usually interesting to practicians), considering that the band of the fluctuation of the signal is usually considerably narrower than the transmission band of the subtracting mechanism we obtain, analogous to (4.11.44),

$$q \approx \frac{\Phi^{-1}(1-F)+\Phi^{-1}(D)}{\sqrt{n}} \frac{\sqrt{\varkappa_{2n}'}}{|H(i0)|^2}, \qquad (4.11.50)$$

$$\varkappa_{2n}' = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(i\lambda)|^4 S_0^2(\lambda)\,d\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} |1-e^{-i\lambda}|^{4m} S_0^2(\lambda)\,d\lambda \qquad (4.11.51)$$

($n\varkappa_{2n}'$ is dispersion of interference at input of relay).

In the case of slow fluctuations of signal reflected from the target, we can find the solution of equation (4.11.9) in the form of the sum of the solution in the presence of one interference and a certain constant. The expression for characteristic function $\Psi(\eta)$ has a form, analogous to (4.4.18):

$$\Psi(\eta) = \frac{1-i\eta}{1-i\eta(1+q_1)} \exp\left\{-\frac{n}{2\pi}\int_{-\pi}^{\pi} \ln\left[1 - i\eta\frac{|H(i\lambda)|^2 S_0(\lambda)}{|H(i0)|^2 S_0(0)}\right] d\lambda\right\}, \qquad (4.11.52)$$

where

$$q_1 = q\frac{n}{S_0(0)}.$$

Considering distribution during the absence of signal from target (q = 0) to be normal and producing convolution of this distribution with a distribution corresponding to the coefficient before the exponent in (4.11.52), we obtain with $1-D \ll 1$ and $1-D \gg F$

$$D \approx \exp\left\{-\frac{\Phi^{-1}(1-F)\sqrt{n\varkappa_{2n}'}}{|H(i0)|^2 S_0(0)\left[1+q\frac{n}{S_0(0)}\right]}\right\},$$

whence, for the threshold signal-to-interference ratio of q, considering that $q\frac{n}{S_0(0)} \gg 1$

when $1 - D \ll 1$, we have

$$q \approx \frac{\sqrt{\varkappa'_{2n}}}{|H(i\theta)|^2} \frac{\Phi^{-1}(1-F)}{\sqrt{n} \ln \frac{1}{D}}. \qquad (4.11.53)$$

Of significant interest is the dependence of the threshold signal-to-interference ratio on multiplicity of subtraction. From (4.11.50) and (4.11.53) it is clear that this dependence is characterized by the ratio

$$\Gamma_1 = \frac{\sqrt{\varkappa'_{2m}}}{|H(i\theta)|^2} = \frac{\sqrt{\sum_{\nu=-2m}^{2m} \binom{4m}{2m-\nu} (-1)^\nu C_\nu}}{\left(2 \sin \frac{\theta}{2}\right)^{2m}}, \qquad (4.11.54)$$

where

$$C_\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_0^2(\lambda) e^{-i\lambda\nu} d\lambda \qquad (4.11.55)$$

are expansion coefficients of function $S_0^2(\lambda)$ in Fourier series at interval $(-\pi, \pi)$ [to obtain formula (4.11.54) it is sufficient to use binomial expansion in (4.11.51)].

In particular, for exponential function of correlation of interference

$$C_\nu = \left(\frac{1+a^2}{1-a^2} + |\nu|\right) \frac{a^{|\nu|}}{(1-a^2)^2}, \qquad (4.11.56)$$

and for function of correlation of form (4.11.24)

$$C_\nu = \frac{I_\nu(2a)}{I_0^2(a)}. \qquad (4.11.57)$$

Calculations by formula (4.11.54) are connected with essential difficulties, since when $\Delta f_0 T_r \ll 1$ a small sum under the radical is received as a result of addition and subtraction of large magnitudes, which in connection with this it is necessary to calculate with high accuracy (to 5 — 6 places and more). When m is sufficiently large, and C_ν can decrease almost to zero, and $\binom{4m}{2m-\nu} \approx \binom{4m}{2m}$, it is possible to extend in (4.11.54) summation ad infinitum, replacing $\binom{4m}{2m-\nu}$ by $\binom{4m}{2m}$. Using Stirling's formula, we obtain

$$\Gamma_1 \approx \frac{1}{\sin^{2m} \frac{\theta}{2}} \cdot \frac{S_0(\pi)}{\sqrt[4]{2\pi m}}. \qquad (4.11.58)$$

As can be seen from this formula, during large m and when $\theta = (2k+1)\pi$ the threshold signal-to-interference ratio very slowly decreases with an increase in multiplicity of the subtraction, and when $\theta = 2k\pi$ an increase of m, on the other hand,

leads to growth of threshold q. It follows from this that it is inexpedient to increase multiplicity of subtraction above that determined.

In order to imagine dependence $\Gamma_1(m)$ during small m and simultaneously to estimate error of approximate formula (4.11.58), let us consider more specifically particular cases (4.11.56) and (4.11.57). In Table 4.3 and 4.4 are presented dependences $\Gamma_1(m)$ for these spectral densities, calculated by exact (4.11.54) and approximate (4.11.58) formulas when $\theta=(2k+1)\pi$. During calculation, the correlation factors between neighboring periods were taken as 0.82. From Table 4.3 it is clear that in a case of exponential function of correlation of interference, the coincidence of exact and approximate formulas is satisfactory for single subtraction and further increase of multiplicity gives only a small gain. For spectral density of the form (4.11.24), which for the considered value a = 3 is near to Gaussian, an increase of multiplicity up to 3 -- 4 is accompanied by a fast decrease of the threshold signal.

Table 4.3.

	Γ_1	m				
		0	1	2	3	4
(a)	(4.11.54) — точная формула	2,25	0,09	0,06	0,05	0,046
(b)	(4.11.58) — приближенная формула		0,063	0,054	0,048	0,045

KEY: (a) Exact formula; (b) Approximate formula.

Table 4.4.

	Γ_1	m				
		0	1	2	3	4
(a)	(4.11.54) — точная формула	1,8	0,12	0,037	0,013	0,0097
(b)	(4.11.58) — приближенная формула		0,0063	0,0054	0,0048	0,0045

KEY: (a) Exact formula; (b) Approximate formula.

The coincidence of the exact and approximate formulas is possible to consider satisfactory also only for $m > 3$, but when $m = 1$ we obtain an error of 20 times.

From the considered examples it is clear that the effectiveness of an increase in multiplicity of subtraction considerably depends on the rate of drop and effective width of the spectrum of fluctuation of the interference. It is possible, apparently, to mention the existence of some threshold multiplicity of subtraction, starting from which a further increase in multiplicity has little effect and is even harmful, considering impairment of the range of the PPC system with respect to speed; and besides this multiplicity m_0 turns out to be smaller for steeply drooping spectral densities. To determine the value of threshold multiplicity it is possible to use the following approximate formula, ensuing from conditions of transition from (4.11.54) to (4.11.58):

$$C_{\sqrt{m_0}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0^2(\lambda) e^{-i\sqrt{m_0}\lambda} d\lambda \sim 0,2. \qquad (4.11.59)$$

We will compare now the effectiveness of the PPC system with the effectiveness of the optimum system. We will use formula (4.11.58), considering multiplicity of m sufficiently great and the possibilities of increasing noise-resistance, owing to an increase of multiplicity, to be exhausted. With this, the relation of threshold values of q, determined by formulas (4.11.50), (4.11.53), (4.11.40), and (4.11.44), are written, with fast fluctuation of the target, in the form

$$\Gamma \approx \frac{1}{\sqrt{\Delta f_1 T_1}} \cdot \frac{\Gamma_1}{S_0(0)} \qquad (4.11.60)$$

and with slow fluctuation

$$\Gamma \approx \frac{\sqrt{n}\,\Phi^{-1}(1-F)}{\ln\frac{1}{F}} \cdot \frac{\Gamma_1}{S_0(0)}. \qquad (4.11.61)$$

Coefficients during $\frac{\Gamma_1}{S_0(0)}$ in both formulas approximately coincide, as we will see in Chapter 5, with the ratio of threshold q for incoherent and coherent accumulation of signal in noise. The presence of interference and multiplicity of subtraction

is fully considered by the factor:

$$\Gamma_2 = \frac{\Gamma_1}{S_0(\theta)} \approx \frac{1}{\sqrt{2\pi m}} \frac{S_0(\pi)}{S_0(\theta)\sin^{2m}\frac{\theta}{2}}. \qquad (4.11.61')$$

When $\theta = \pi$

$$\Gamma_2 \approx (2\pi m)^{-\frac{1}{4}}.$$

Subtraction in this case somewhat decreases the loss connected with incoherent accumulation of signal; however, very insignificantly. During other θ the selectivity of the PPC system shows up: $\Gamma_2 \to \infty$ when $\theta \to 0$.

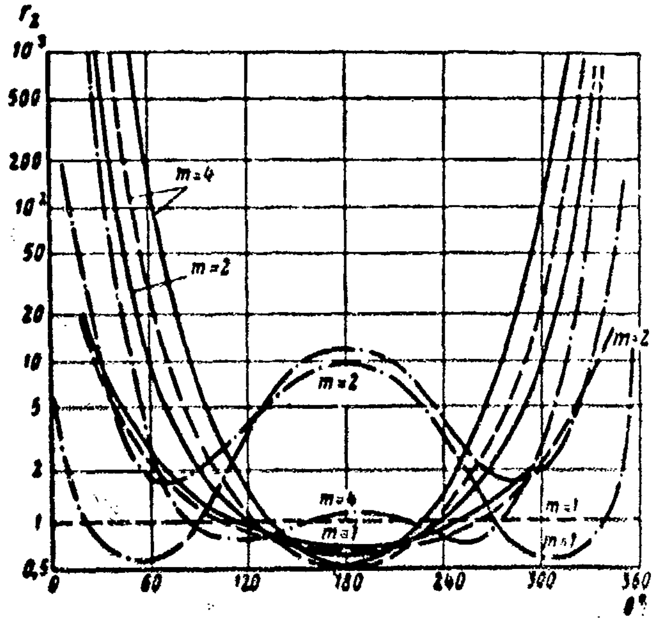

Fig. 4.29. Dependence of Γ_2 on θ: ——— $S_0(\theta) = 1$; — — — — $S_0(\theta) = 0.2/(1 - \cos\theta)$; —·—·— $S_0(\theta) = 0.2e^{\cos\theta}$.

Fig. 4.29 presents dependence of Γ_2 on θ during various m for the above considered examples (Tables 4.3, 4.4) and for interference equivalent to white noise $S_0(\theta) = 1$. The character of these dependences shows that near blind speeds the loss of the system of period-by-period compensation is considerably increased.

In conclusion we will pause shortly on the question of comparing two-channel and single-channel PPC systems. It is possible to show that during the use of one channel the characteristic function $\Psi(\eta)$ is equal to the square root of the characteristic function corresponding to two-channel processing. Owing to this, n in the final formulas decreases twice, and in (4.11.53) lnD is replaced by function $K_1^{-1}(D)$ [see

(4.4.14)]. As a result, the gain of a two-channel system during slow fluctuation of signal from target is determined by the ratio

$$\frac{\sqrt{2}\ln\frac{1}{D}}{K_1^{-1}(D)},$$

and during fast fluctuation is near to $\sqrt{2}$.

4.11.4. Effectiveness of the PPC System with External Coherence

For a system of period-by-period compensation with external coherence with those same assumptions concerning multiplicity of subtraction, the output signal also can be represented in the form (4.11.48), where instead of f_j enters $|f_j|^2$, if we consider the detector quadratic. Magnitude L no longer is a quadratic form of normally distributed magnitudes, and the characteristic function of this magnitude cannot be found. In connection with this we will limit ourselves to consideration of a case of fast fluctuation of signal from target and interference. With this, it is possible approximately to consider magnitude L distributed by normal law and to limit ourselves to calculation of its mean value and dispersion:

$$\varkappa_1 = \bar{L} = \sum_{j,k=1}^{n} v_{jk}\overline{|f_j f_k|^2}, \tag{4.11.62}$$

$$\varkappa_2 = \overline{L^2} - \bar{L}^2 = \sum_{j,k,\mu,\nu} v_{jk}v_{\mu\nu}\left(\overline{|f_j f_k f_\mu f_\nu|^2} - \overline{|f_j f_k|^2}\,\overline{|f_\mu f_\nu|^2}\right). \tag{4.11.63}$$

Considering that the real and imaginary parts of f_j are distributed by normal law, and using the expression for fourth mixed moments of normal distribution, it is possible to show that

$$\overline{|f_j f_k|^2} = \overline{|f_j|^2}\,\overline{|f_k|^2} + \overline{f_j f_k}\,\overline{f_j^* f_k^*} + \overline{f_j f_k^*}\,\overline{f_j^* f_k}. \tag{4.11.64}$$

Substituting (4.11.64) into (4.11.62) we obtain a difference of mean values in the presence of and during the absence of a signal from a target, in equation (4.4.10) of detection characteristics in the form

$$\varkappa_1 - \varkappa_{10} = q\sum_{j,k=1}^{n} v_{jk}\{q[1+\rho^2(j,k)] + 2[1+\rho(j,k)r_0(j,k)\cos(j-k)\theta]\}. \tag{4.11.65}$$

Dispersion $\varkappa_2$ is expressed through eighth mixed moments of normal distribution and its calculation is connected with an unusually awkward transformation (108 addends must be written out and grouped), as a result of which we obtain

$$\varkappa_{2n} = 2 \sum_{j, k, \mu, \nu} v_{jk} v_{\mu\nu} \{2r_0^2(j, \mu) + r_0^2(j, \mu) r_0^2(k, \nu) + \tag{4.11.66}$$

$$+ 4r_0(j, k) r_0(j, \mu) r_0(k, \nu) + 2r_0(j, k) r_0(j, \mu) r_0(k, \nu) r_0(\mu, \nu) +$$

$$+ r_0(j, \mu) r_0(j, \nu) r_0(k, \mu) r_0(k, \nu)\}.$$

Dispersion in the presence of useful signal is expressed in an even more complicated manner. Therefore, we will limit ourselves to the calculation of threshold q, considering $\varkappa_2 \simeq \varkappa_{2n}$.

Let us consider the simplest case of exponential correlation of interference and single subtraction $(v_{jj} = 2,\ v_{j\,j-1} = v_{j-1\,j} = -1,$ the remaining $v_{jk} = 0)$. In this case, all calculations are brought to an end and the expression for q has the form

$$q \simeq \frac{\Phi^{-1}(1 - F) + \Phi^{-1}(D)}{\sqrt{n}} \frac{\sqrt{\varkappa'_{2n}}}{2(1 - \rho r \cos\theta)}, \tag{4.11.67}$$

where $\rho = e^{-2\Delta f_c T_r}$ and $r = e^{-2\Delta f_n T_r}$ are coefficients of correlation of neighboring periods of signal and interference, and

$$\varkappa'_{2n} = \frac{9 - 20r^2 + 10r^4 + 4r^6 - 3r^8}{1 - r^4}.$$

Fig. 4.30 shows the dependence of ratio Γ of the threshold values of q for systems of period-by-period compensation with internal and external coherence on $\Delta f_n T_r$ when $\theta = (2k + 1)\pi$ and various $\Delta f_c T_r$. As can be seen from the figure, the effectiveness of a system with external coherence is significantly lower. During $\Delta f_n T_r \to 0$ ratio Γ is limitlessly increased as $\frac{1}{\sqrt{2\Delta f_n T_r}}$. For systems with higher multiplicity of subtraction and steeply drooping spectra, the advantages of a system with internal coherence will be, apparently, still more convincing.

Comparatively low effectiveness is not the only deficiency of a system with external coherence. Another important deficiency is the absence at the output of the receiver of signals from the target and from passive interferences, if these signals do not overlap.

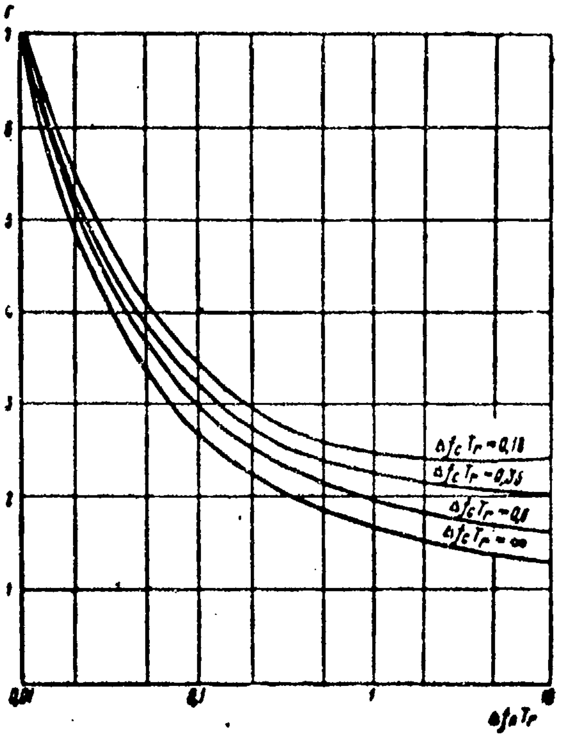

Fig. 4.30. Loss in the threshold signal-to-interference ratio for a PPC system with external coherence.

4.11.5. Problem of Blind Speeds. Wobbling of Repetition Period

As was shown above, the sure selection of a signal on a background of passive interferences is impossible during so-called blind speeds, when spectral lines of the signal and interference coincide. The values of blind speeds are determined, as is easily seen, by formula

$$v_k = \frac{k\lambda_0}{2T_r}, \qquad k = 0, \pm 1, \ldots,$$

where λ_0 is wave length.

With a large repetition period, blind speeds are so frequent that some of them must occur in the speed range of the targets, on which a given radar must operate. If abrubt decrease of T_r is impermissible, it is necessary to use other methods of removal of blind speeds from the working speed range. The essence of these methods, as can be seen from the above mentioned formula, inevitably leads to a change in the time of the repetition period [59] or to the use of several signals, delivered on

a frequency, for which blind speeds do not coincide. Retuning of carrier frequency cannot be used in practice, since the retuning necessary for an essential change of blind speeds leads to a decrease, practically to zero, of the correlation of interfering reflections, corresponding to various frequencies.

In this division will be considered a system with wobbling of period. During a variable period, functions of correlation of interference and signal $\rho(j, k)$ and $r_0(j, k)$ no longer depend only on the difference of numbers of periods (sequence of results of intraperiodic processing becomes nonstationary). We will assume that this nonstationary nature is manifested only at a magnitude of phase shift ϑ, so that

$$R_{jk}=\overline{f_j f_k^*}=q\rho(j, k)e^{-i\sum_{\nu=j+1}^{k}\vartheta_\nu}+r_0(j, k),$$

where $\rho(j, k)$ and $r_0(j, k)$ are the same function of correlation as in the case of constant T_r;

ϑ_ν is phase shift in the νth period.

Such an assumption assumes relatively little change of period T_r, which usually takes place in practice. We also consider that wobbling of period is produced according to a certain periodic law and the wobbling period is small as compared with the time of observation and the time of correlation of signal fluctuation.

With the assumptions made, it is possible to show that the diagram (Fig. 4.24), synthesized under certain conditions for the case T_r = const, is optimum and a case of variable period of alternation, if in them multiplication by $\nu\vartheta$ and $\sin\nu\vartheta$, utilized for transfer of signal spectrum to zero frequency before coherent accumulation, is replaced by multiplication by $\cos\sum_{i=1}^{\nu}\vartheta_i$ and $\sin\sum_{i=1}^{\nu}\vartheta_i$. It is obvious that when T_r = var methods of making filters, suppressing interference, and carrying out signal accumulation change. The delay per period, utilized in these filters, should be variable.

Let us consider the detection characteristics of an optimum system for case T_r = var. With slow fluctuation of target, the signal-to-interference ratio of q_1 can be calculated by general formula (4.11.12). When calculating the form of the bunch and changeability of θ we obtain

$$q_1 = \sum_j Z_j g_{j-\nu} e^{-i\sum_1^j \theta_\mu}, \qquad (4.11.68)$$

where $g_{j-\nu}$ is the bunch envelope of pulses. (see 4.11.29).

For the considered case Z_j is expressed by formula

$$Z_j = \sum_{l=-\infty}^{\infty} w(j, l) g_{l-\nu} e^{i\sum_1^l \theta_\mu}. \qquad (4.11.69)$$

Substituting (4.11.69) into (4.11.68) and considering (4.11.28), we find

$$q_1 = \frac{q}{2\pi}\int_{-\pi}^{\pi}\frac{1}{S_0(\lambda)}\left|\sum_{j=-\infty}^{\infty} g_{j-\nu} e^{i\sum_1^j \theta_\mu - ij\lambda}\right|^2 d\lambda. \qquad (4.11.70)$$

Considering the assumption concerning a small change of g_k for wobbling period m_0, it is possible to convert the sum under the integral to the form

$$\sum_{k=-\infty}^{\infty} g_{km_0} e^{-ikm_0(\lambda-\theta)} \sum_{\nu=1}^{m_0} e^{i\sum_1^\nu \theta_\mu - i\nu\lambda}, \qquad (4.11.71)$$

where $\theta = \frac{1}{m_0}\sum_1^{m_0}\theta_\mu$ is the average phase shift.

The first sum in (4.11.71) is $G[m_0(\lambda-\theta)]$ the spectrum of the bunch envelope of pulses. Function $G[m_0(\lambda-\theta)]$ is periodic with respect to λ with period $\frac{2\pi}{m_0}$ and, in view of the great duration of the bunch, diminishes quickly with the increase of $\left|\lambda-\theta-\frac{2\pi k}{m_0}\right|$. Using the theorem concerning the average, instead of (4.11.70), we obtain

$$q_1 \approx q\frac{n_{\text{эф}}}{m_0^2}\sum_{k=1}^{m_0}\frac{1}{S_0\left(\theta+\frac{2\pi k}{m_0}\right)}\left|\sum_{\nu=1}^{m_0} e^{i\sum_1^\nu(\theta_\mu-\theta)-i\nu\frac{2\pi k}{m_0}}\right|^2 = \qquad (4.11.72)$$

$$= q\frac{n_{\text{эф}}}{S_{0\,\text{экв}}(\theta)}.$$

This formula in form coincides with (4.11.40).

For a case of fast fluctuation, calculation of mean value and dispersion of magnitude L in the output of the diagram in Fig. 4.24, modified in the above described form for reception of signals with variable T_r, leads to an expression for threshold q_1, similar to (4.11.44) in which $S_0(\theta)$ is replaced by $S'_{0\,\text{экв}}(\theta)$, where $S'^{\,2}_{0\,\text{экв}}(\theta)$ is expressed by formula (4.11.72) if $S_0(\theta)$ is replaced by $S_0^2(\theta)$. As in the derivation of (4.11.44), during calculation, assumptions are used concerning the fact that $q \ll 1,\ \varkappa'_{2\text{п}} \approx \varkappa'_2,\ \tau_{k\text{п}} \ll \tau_{k0}$.

When $m_0 = 2$ and $\rho(j,\ k) = a^{|j-k|}$

$$S_{0\,\text{экв}}(\theta) = \frac{1 - a^2}{1 + a^2 - 2a\cos\theta\cos\frac{\Delta\theta}{2}}; \qquad (4.11.73)$$

when

$$\rho(j,\ k) = \frac{I_{|j-k|}(a)}{I_0(a)}$$

$$S_{0\,\text{экв}}(\theta) = \frac{1}{I_0(a)\left[\cosh(a\cos\theta) - \cos\frac{\Delta\theta}{2}\sinh(a\cos\theta)\right]}. \qquad (4.11.74)$$

Similar relationships are obtained also for $S'_{0\,\text{экв}}(\theta)$:

$$S'^{\,2}_{0\,\text{экв}}(\theta) = \frac{(1 - a^2)^2}{(1 + a^2)^2 + 4a^2\cos^2\theta - 8a(1 + a^2)\cos\theta\cos\frac{\Delta\theta}{2}}, \qquad (4.11.75)$$

$$S'^{\,2}_{0\,\text{экв}}(\theta) = \frac{1}{I_0^2(a)\left[\cosh(2a\cos\theta) - \cos\frac{\Delta\theta}{2}\sinh(2a\cos\theta)\right]}. \qquad (4.11.76)$$

As can be seen from relationships obtained, the magnitude of threshold q considerably depends on average phase shift θ and difference of phase shifts $\Delta\theta$. Best results are obtained when $\Delta\theta = (2k+1)\pi$, when $S_{0\,\text{экв}}(\theta)$ depends on θ very weakly [in case (4.11.73) not at all, and in case (4.11.74) $S'_{0\,\text{экв}}(\theta)$ is less than $\frac{1}{I_0(a)}$ during all θ]. In accordance with this it is necessary to select the magnitude of shift periods in such a manner that $\Delta\theta = 4\pi\frac{v}{\lambda_0}(T_{r_1} - T_{r_2})$ is near to $(2k+1)\pi$ in the working speed range v. Width Δv of the range in which blind speeds do not affect detection quality depends on allowable increase of $S_{\text{экв}}(\theta)$ as compared with minimum wh. $\Delta\theta = (2k+1)\pi$. It is possible to consider permissible an increase of $\cos\frac{\Delta\theta}{2}$ up to $\frac{1}{2}$. With this, even in the case of very narrow-band interference ($a \approx 1,\ a \gg 1$) $S_{\text{экв}}(\theta)$ during the least

favorable θ turns out to be approximately 2 times more than during $\cos\frac{\Delta\theta}{2}=0$. For width of speed ranges with this we obtain

$$\Delta v=\frac{\lambda_0}{6\Delta T_r},$$

and besides central speeds of ranges $v_k=\frac{k+\frac{1}{2}}{2}\cdot\frac{\lambda_0}{\Delta T_r}$, so that $\Delta v=\frac{2}{3}v_0$. For many practical problems this range turns out to be too narrow, and for satisfying the placed requirements it is necessary to select a number of shift periods larger than two. To the most uniform working speed range of equivalent spectral density $S_{0\text{экв}}(\theta)$ one should, as may be seen from (4.11.72), select value of T_{r_i} in such a manner that the modulus of the sum, in (4.11.72), is approximately identical for all k and changes little in the shown range.

In existing radar systems for reception of signals, wobbulated with respect to repetition period systems of period-by-period compensation are used [59]. Statistical characteristics of interference in the output of such a system, under the assumptions made about small change of period, do not differ in any way from corresponding characteristics during a constant period. To calculate detection characteristics during fast fluctuation we have to calculate the difference of mean values of signal with interference and the interference and dispersion of interference in the output of the system of detection. In accordance with (4.11.48) we obtain

$$\varkappa_1'-\varkappa_{1\text{п}}'=q\sum_{j,k}v_{jk}\rho(j,k)e^{-i\sum_{l=j}^{k}\psi_l}=$$

$$=\frac{q}{2\pi}\int_{-\infty}^{\infty}|H(i\lambda)|^2\sum_{j,k}\rho(j,k)e^{-i\sum_{l=j}^{k}\psi_l+i(j-k)\lambda}\,d\lambda.$$

Considering $\rho(j,k)$ as changing so slowly that $\rho(j+m_0,k)\approx\rho(j,k)$, it is possible to convert this expression to the form

$$\varkappa_1-\varkappa_{1\text{п}}\approx\frac{q}{m_0}\sum_{k=1}^{m_0}\left|H\left(i\theta+i\frac{2\pi k}{m_0}\right)\right|^2\left|\sum_{r=1}^{m_0}e^{i\sum_{1}^{r}\left(\psi_r-\theta-\frac{2\pi k}{m_0}\right)}\right|^2=\tag{4.11.77}$$

$$=q\,|H_{\text{экв}}(i\theta)|^2.$$

Assuming the threshold q to be sufficiently small so that $\varkappa_2\approx\varkappa_{2\text{п}}$, and using (4.4.10), we obtain for a two-channel circuit of compensation

$$q\approx\frac{\Phi^{-1}(1-F)+\Phi^{-1}(D)}{\sqrt{\pi}}\,\frac{\sqrt{\overline{\varkappa_{2\text{п}}}}}{|H_{\text{экв}}(i\theta)|^2}.\tag{4.11.78}$$

This formuls is completely analogous to (4.11.50), received for T_r = const, with the only difference that $H(i\theta)$ is replaced by $H_{\text{экв}}(i\theta)$.

To determine the threshold q in the case of slow fluctuation it is necessary to solve equation (4.11.9) in the presence of a signal for a case of variable T_r. Making all necessary calculations, we manage to show that during $1-D \gg F$ and $1-D \ll 1$ threshold q is expressed by formula (4.11.53), where $|H(i\theta)|^2$ should be replaced by $|H_{\text{экв}}(i\theta)|^2$. Thus, equivalent frequency response $|H_{\text{экв}}(i\theta)|^2$ completely determines the dependence of threshold q on θ.

When $m_0 = 2$ and single subtraction

$$|H_{\text{экв}}(i\theta)|^2 = 2\left(1 - \cos\frac{\Delta\theta}{2}\cos\theta\right).$$

Best results in the given case, as for an optimum circuit, are obtained when $\Delta\theta = (2k+1)\pi$. Obviously, also in force are formulas relative to the width of the speed range. When $m_0 > 2$ the uniformity of frequency response is determined by the behavior of the square of the modulus of the sum in (4.11.72) and (4.11.77).

The magnitude of loss of a PPC system in comparison with an optimum circuit is determined, as is easily seen, by the same formulas as in a case of constant T_r when replacing $S_0(\theta)$ and $|H(i\theta)|^2$ by $S_{\text{экв}}(\theta)$ and $|H_{\text{экв}}(i\theta)|^2$.

4.11.6 The Problem of Blind Speeds. The Use of Several Frequency Channels

As already noted, to combat blind speeds we can also use simultaneous operation on several (for example, two) carrier frequencies chosen so that the blind speeds corresponding to these frequencies do not coincide in the working speed range. We will assume that separation of frequencies is sufficiently great that reflection of signals from target and from passive interferences, corresponding to various carrier frequencies, can be considered statistically independent (below this assumption is well-grounded). With this, as was shown in Section 4.1, optimum operations consist in optimum processing of each of the signals separately with subsequent addition of

the results of processing. The circuit of an optimum receiver consists of several channels, built according to the diagram of Section 4.24. The number of channels is equal to the number of working frequencies. Outputs of channels are summarized and are compared with threshold. In the case of fast fluctuation of signal, the summation can be carried out directly after forming the square of the envelope of the filtered signal. With this, for incoherent accumualtion is used one accumulator common for all channels.

Let us consider detection characteristics corresponding to optimum processing. With this, we will consider that the statistical properties of signals on all frequencies are identical. During fast fluctuation of signal and interference, semi-invariants of distribution can be found by summation of semi-invariants for separate channels, expressed by formulas (4.11.42). Using these semi-invariants, it is possible to examine the detection characteristics using either formula (4.4.10), if n is sufficiently great, or expansion (4.4.11).

Let us consider the most interesting case for the practician when the threshold value of q is small. With this, assuming the time of correlation of interference to be small as compared with the time of correlation of signal, we obtain similarly to (4.11.44)

$$q \approx \frac{\Phi^{-1}(1-F)+\Phi^{-1}(D)}{\sqrt{\frac{n}{2\pi}\int\limits_{-\infty}^{\infty}\sum\limits_{j=1}^{m_0}\frac{S^2(\lambda-\theta_j)}{S_0^2(\lambda)}\,d\lambda}} \approx \frac{\Phi^{-1}(1-F)+\Phi^{-1}(D)}{n\sqrt{\sum\limits_{j=1}^{m_0}\frac{1}{S_0^2(\theta_j)}}}\sqrt{\Delta f_1 T}, \tag{4.11.79}$$

where θ_j are the phase shifts for period, corresponding to various frequencies.

Let us note that magnitude q in (4.11.79) represents the signal-to-interference ratio in one channel, determined by the power in this channel and the level of noises in it. If noises can be disregarded, then the signal-to-interference ratio does not depend on radiated power and is equal to the same magnitude for a separate

frequency channel and for a combination of channels.

Threshold q depends on magnitudes of phase shifts through magnitude

$$\sum_{j=1}^{m_0} \frac{1}{S_0^2(\theta_j)} = \frac{1}{S^2_{\text{экв}}}.$$

Radiated frequencies should be selected so that this sum has the largest possible magnitude in the working speed range.

In case $m_0 = 2$ we obtain

when $r_0(j, k) = \alpha^{|j-k|}$

$$S^2_{\text{экв}} = \frac{1 - \alpha^2}{2(1+\alpha^2)^2 + 4\alpha^2\left[\cos^2\left(\theta + \frac{\Delta\theta}{2}\right) + \cos^2\left(\theta - \frac{\Delta\theta}{2}\right)\right] - 8\alpha(1+\alpha^2)\cos\theta\cos\frac{\Delta\theta}{2}}, \tag{4.11.80}$$

when $S_0(\theta) \sim e^{a\cos\theta}$

$$S^2_{\text{экв}} = \frac{1}{I_0^2(a)\left[e^{-2a\cos\left(\theta + \frac{\Delta\theta}{2}\right)} + e^{-2a\cos\left(\theta - \frac{\Delta\theta}{2}\right)}\right]}. \tag{4.11.81}$$

The lowest value of equivalent spectral density of the interference in both cases is obtained when $\Delta\theta = (2k+1)\pi$. With this, $S_{\text{экв}}$ turns out to be $\sqrt{2}$ times less than spectral density $S'_{0\text{экв}}(\theta)$, expressed by formulas (4.11.75) and (4.11.76).

Thus, during fast fluctuation and $\Delta\theta$, near to $(2k+1)\pi$, the effectiveness of a two-frequency system turns out to be higher than a system with two shift periods. This is explained by the fact that, differing from detection on a background of noises, weakening of fluctuation owing to multi-frequency is not accompanied by a decrease of the signal-to-interference ratio in each frequency channel, inasmuch as the power of the signal and interference equally depend on power radiated on each channel.

During slow fluctuation, the distributive law for the sum of output voltages of frequency channels is convolution m_0 of exponential distributions. In the absence of useful signal, statistical characteristics of components are identical and for

a sum is obtained chi-square distribution with $2m_0$ degrees of freedom. In accordance with this, the probability of false alarm is

$$F = K_{2m}(c) = \int_c^\infty \frac{x^{m-1} e^{-\frac{x}{2}}}{2^m (m-1)!} dx, \qquad (4.11.82)$$

where c is the relation of the threshold of operation to the dispersion of interference at output of the storing filter.

In the presence of the signal, distributions of components are equal because of the difference in phase shifts θ_j. Characteristic function of the sum can be represented in this case in the form of (4.6.1). The corresponding equation of detection characteristics coincides with (4.6.2), if q_{0j} is replaced by the signal-to-interference ratio in the j-th channel q_{1j}.

Let us consider more specifically the case of two working frequencies. With this,

$$D = \frac{1}{q_{11} - q_{12}} \left[(1 + q_{11}) e^{-\frac{K_4^{-1}(F)}{2(1+q_{11})}} + (1 + q_{22}) e^{-\frac{K_4^{-1}(F)}{2(1+q_{11})}} \right]. \qquad (4.11.83)$$

Dependence q_{11} and q_{12} on θ_1 and θ_2 is determined by formulas in paragraph 4.11.2. In particular, assuming the time of correlation of interference to be small in comparison with the duration of the bunch, it is possible to calculate q_1 by formula (4.11.40).

Fig. 4.31 presents the dependence of threshold q on $\Delta\theta = \theta_1 - \theta_2$ for a case of exponential correlation of interference when D = 0.9, n = 25, F = 10^{-4}, $r = 0.9$. Inasmuch as q depends on θ, in the figure there is a shaded region, in which q can change at a given $\Delta\theta$. For comparison, on the same graph, the dependence of threshold q on $\Delta\theta$ is presented for a system with two shift periods with the same values of all parameters. The smallest q is obtained when $\Delta\theta = (2k+1)\pi$. With this, differing from a case of shift periods, q continues to depend on θ.

Comparison of results for a two-frequency diagram and a diagram with shift periods shows that during slow fluctuation the two-frequency diagram possesses

noticeable advantages which are the result of a relative decrease of target fluctuation during the use of two independently fluctuating signals. A similar effect was noted above during fast fluctuation; however, there it was less. Range, with respect to speed, as one may see from the figure, also can be determined from condition $\cos\frac{\Delta\theta}{2}=\pm 1/2$. With this, q increases not more than twice, as compared with its value when $\Delta\theta=(2k+1)\pi$. Speed ranges, in which the influence of blind zones is removed, are concentrated nearby

$$v_k=\left(k+\frac{1}{2}\right)\frac{c}{2\Delta f T_r},$$

where Δf is the difference of carrier frequencies, and they have the width

$$\Delta v=\frac{1}{6}\frac{c}{\Delta f T_r}=\frac{2}{3}v_0.$$

It is necessary to note that the tendency to remove blind speeds in the working range along with the considerations presented in Section 4.6 also leads to the use

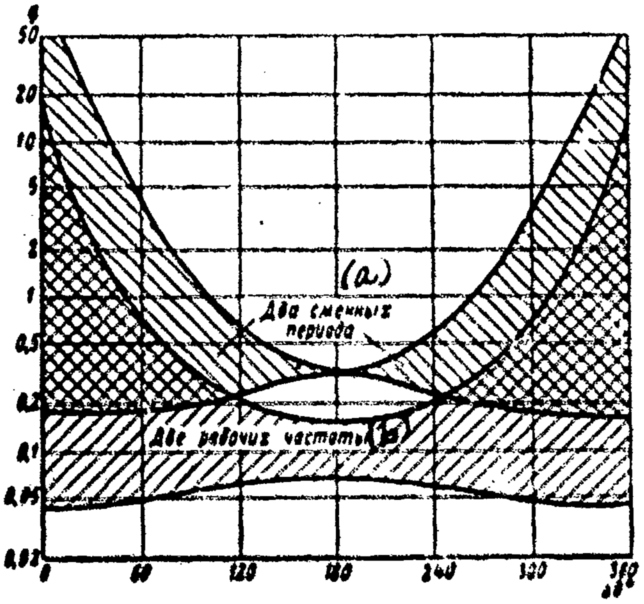

Fig. 4.31. Dependence of threshold q on Δθ for a system with two shift periods and two working frequencies.
KEY: (a) Two shift periods; (b) Two working frequencies.

of frequency channels detuned to a magnitude sufficient for statistical independence of corresponding reflected signals. As can be seen from the above mentioned formulas, in order to combine first optimum speed with the center v of the working range,

detuning is necessary:

$$\Delta f_{\text{мин}} \approx \frac{1}{4}\frac{c}{vT_r}.$$

For example, when $v = 500$ m/sec and $T_r = 10^{-3}$ sec, detuning is $\Delta f_{\text{мин}} \approx 200$ megacycles. Such detuning is fully sufficient for statistical independence of signals from aircraft and all the more so for interfering reflections.

Above have been analyzed detection characteristics for a case, when the signal in each frequency channel is processed in an optimum manner. We will see what the consequences will be if we replace optimum processing with period-by-period subtraction. With this, for simplicity we will limit ourselves to the case of fast fluctuation, when it is possible to consider distribution at input of relay to be normal. For $x'_1 - x'_{1ш}$ and $x'_{1ш}$ we have

$$x'_1 - x'_{1ш} = \frac{q}{2\pi}\int_{-\infty}^{\infty}|H(i\lambda)|^2\sum_{j=1}^{m_0}S(\lambda-\theta_j)\,d\lambda \approx q\sum_{j=1}^{m_0}|H(i\theta_j)|^2, \qquad (4.11.84)$$

$$x'_{1ш} = \frac{m_0}{2\pi}\int_{-\infty}^{\infty}|H(i\lambda)|^2 S_0^2(\lambda)\,d\lambda. \qquad (4.11.85)$$

Using (4.11.84) and (4.11.85) considering q to be small, we can find an expression for threshold q in a similar manner as was done for an optimum system. The result of calculation is presented graphically in Fig. 4.32 for a case of single subtraction and exponential functions of correlation of interference and signal when $D = 0.9$, $F = 10^{-4}$, $r = 0.8$, $\rho = 0.9$, $n = 50$/ $m_0 = 2$ in the form of dependence of q on $\Delta\theta$. In the figure there is a shaded region, in which q can be changed during a given $\Delta\theta$. For comparison in the same place is shown a similar dependence for an optimum system with the same values of parameters.

For a case of two carrier frequencies, as a natural further simplification of optimum processing, let us consider a variant with unification of frequency channels before subtraction. Such unification can be attained by means of the mixing of signals spaced by frequency. With this, further processing should be carried out on

a difference frequency. Let us consider the case when this processing is period-by-period subtraction in two quadrature channels. With this, as is easy to see,

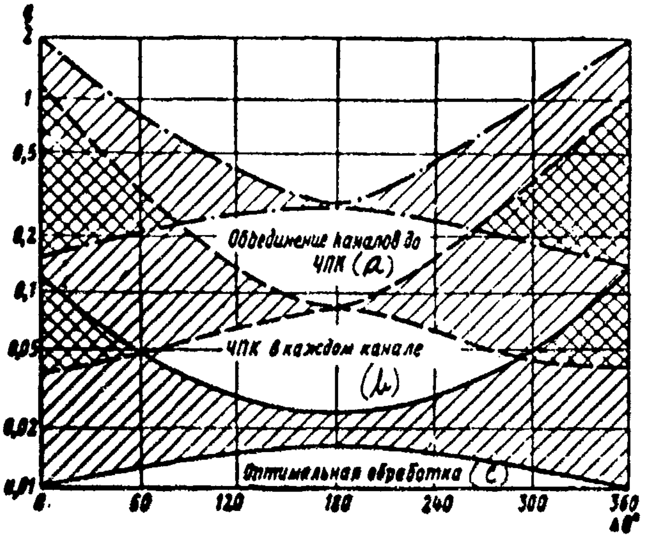

Fig. 4.32. Dependence of threshold q on $\Delta\theta$ for three methods of processing a two-frequency signal.
KEY: (a) Unification of channels before PPC: (b) PPC in each channel; (c) Optimum processing.

signals at input of system of subtraction can be represented in the form

$$\varphi'_j = \frac{1}{2}[f_{1j}f^*_{2j} + f^*_{1j}f_{2j}] = \mathrm{Re}\, f_{1j}f^*_{2j},$$

$$\psi'_j = \frac{1}{2}[f^*_{1j}f_{2j} - f_{1j}f^*_{2j}] = \mathrm{Im}\, f_{1j}f^*_{2j},$$

where f_{1j} and f_{2j} are results of intraperiodic processing of signals in the first and second frequency channels.

The accumulated signal is the sum of two magnitudes:

$$L' = \sum_{j,k} v_{jk}\varphi'_j\varphi'_k + \sum_{j,k} v_{jk}\psi'_j\psi'_k. \qquad (4.11.86)$$

The possibility of such a signal concept has already been discussed in paragraph 4.11.3, where the expression for v_{jk} is introduced through the frequency characteristics of the system of subtraction. Using the symmetry of coefficients v_{jk}, it is easy to show that (4.11.86) can be written in the form

$$L' = \sum_{j,k} v_{jk} f_{1j} f^*_{2j} f_{1k} f^*_{2k}. \qquad (4.11.86')$$

We cannot find a distributive law for L' in the general case. We will limit ourselves to the consideration of a case of fast fluctuation, when this law can be considered normal. Considering the independence of magnitudes f_{1j} and f_{2j}, we obtain

$$\varkappa_1 - \varkappa_{1\text{п}} = q\sum_{j,k} v_{jk}\Big[q\rho^2(j,k)e^{i(j-k)\Delta\theta} + 2\rho(j,k)r_0(j,k)e^{i(j-k)\theta}\cos\frac{\Delta\theta}{2}\Big]. \qquad (4.11.87)$$

$$\varkappa_{2\text{п}} = \sum_{j,k,\mu,\nu} v_{jk}v_{\mu\nu}[2r_0(j,k)r_0(\mu,\nu)r_0(j,\mu)\times r_0(k,\nu) + r_0^2(j,\mu)r_0^2(k,\nu)]. \qquad (4.11.88)$$

In a particular case of single subtraction and exponential functions of correlation of **interference and signal**

$$\varkappa_1' - \varkappa_{1\text{п}}' = 2q\Big[q(1-\rho^2\cos\Delta\theta) + 2\rho r\Big(1-\cos\theta\cos\frac{\Delta\theta}{2}\Big)\Big]. \qquad (4.11.89)$$

$$\varkappa_{2\text{п}}' = \frac{14-28r^2+8r^4+12r^6-6r^8}{1-r_0^4}. \qquad (4.11.90)$$

Fig. 4.32 shows the dependence q on $\Delta\theta$, calculated on the basis of these formulas with the same values of initial parameters as the other curves of this figure. As can be seen from the graph, threshold q when $\Delta\theta = (2k+1)\pi$ turns out to be, for the given method of processing, approximately 10 times more than for optimum processing, and approximately 3 times more than for a system with subtraction before unification of channels. Such a sharp increase in the threshold signal-to-interference ratio is connected mainly with expansion of the interference spectrum during the mixing of reflected signals, decrasing the effectiveness of subsequent processing.

4.12. The Effect of Active Interferences on Systems of Detection of Coherent Signal

In this paragraph let us consider the question of the effect of certain, most widespread forms of active interferences (noise, random pulse and relay) on systems of radar detection of target. The problem of this consideration is the quantitative appraisal of the effectiveness of the effect of various interferences.

4.12.1. Noise Jamming Interference

Noise jamming interference is a broadband random process, whose distributive law depends on the method of creating interference and can differ considerably from normal. However, under the effect of this process on the narrowband mechanisms in a number of elements of the system of detection of coherent signal, there occurs a mutual imposition of large number of independent values of this process, distant from each other in time at intervals, larger than the time of correlation, but smaller than the time constant of the given mechanism. With this, due to the central limit theorem, the process on output approaches normal. Therefore, if the spectrum of interference within limits of the width of the modulation spectrum can be considered uniform, the effect of interference on the detection system does not differ in character from the effect of Gaussian white noise with equivalent spectral density $N_{\text{п}}$. In the equation of detection characteristics instead of the signal-to-noise ratio we will enter the signal-to-interference ratio:

$$q'_{\text{сп}} = \frac{P_{\text{с}} T}{2(N_0 + N_{\text{п}})}.$$

Therefore, the reliability of detection with the same probability of false alarm is considerably lowered.

The increase in intensity of noise during a constant level of operation of the relay leads also to an increase in frequency of false alarm in the system of detection. In order to avoid an increase in this frequency above that permissible, usually in the receiver is introduced a system of automatic noise gain control (ANGC), ensuring approximate constancy of noise level in the output of the receiving channel. In systems of visual detection ANGC is frequently replaced by hand adjustment. The signal directing ANGC is usually taken from a section of distances or frequencies, in which the presence of a target is excluded. An ANGC system is constructed usually just as a system of automatic gain control, working on a signal from target. We

will idealize the work of ANGC, considering that this system ensures the bringing of interference and noises to a level equal to the level of natural receiver noises. With this, an increase in the frequency of false alarms does not occur and the effect of interference is considered by the corresponding change in the signal-to-noise ratio. The range of the radar in the presence of interference is expressed through distance corresponding to the same probabilities of F and D during the absence of interference, in the following manner:

$$d_1 = \frac{d_0}{\sqrt[4]{1+\frac{N_{\text{п}}}{N_0}}} = \frac{d_0}{\sqrt[4]{1+\frac{N_{\text{изл}} G_{\text{п}} G(\varphi_{\text{п}}, \theta_{\text{п}}) \lambda^2}{16\pi^2 d_{\text{п}}^2 k \vartheta_{\text{ш}} N}}}. \qquad (4.12.1)$$

where $N_{\text{изл}}$ is the spectral density of the radiating power of the jamming transmitter in the receiver band;

$G_{\text{п}}$ is the antenna gain of the station of interferences;

$d_{\text{п}}$ is the distance to jamming transmitter;

$G(\varphi_{\text{п}}, \theta_{\text{п}})$ is the directivity factor of the radar antenna in the direction of the jamming transmitter;

λ is the wave length;

k is the Boltzmann constant;

N is the absolute temperature of antenna;

$\vartheta_{\text{ш}}$ is the noise factor in the receiving device.

If the problem of the radar set is the determination of the direction to the source of interference, then the detection system should accomplish the capture of interference on angles. With this, inasmuch as the level of interference is unknown, capture can occur and tracking of interference start, effective in the direction of the parasitic lobe. To combat this phenomenon it is possible to use ANGC with a limited dynamic range, selected or regulated with respect to interference in such a manner that the interference received by the parasitic lobe is processed well by the ANGC system, and during the hit of interference in the main lobe, control is stopped and operation of the relay takes place. This method is useful only in a

limited range of interference levels, and its effectiveness will be greatly weakened with a variable level of interferences.

A more radical method of protection [59] consists in the use, along with a signal recieved on the main antenna, of a signal from an additional antenna, for which the amplitude diagram coincides with the form of parasitic lobes of the diagram of the main antenna (or the corresponding adjustment of amplification factors is used). As a result of joint processing of signals from the output of both antennas, interference effective in the parasitic lobe is partially compensated.

Let us consider the optimum method of processing the signals received and we will estimate the potential noise-resistance of the suggested method. For synthesis of an optimum system in this case, assuming the signal and interference to be normal random processes, it is possible to use relationships (4.2.9) -- (4.2.11). Interference, in accordance with the above, will be replaced by white noise. The signal from the target, received by the additional antenna, will be disregarded. With this, for functions of correlation of signals $y_1(t)$ and $y_2(t)$ received on the main and additional channels we have, respectively,

$$R_{11}(t_1, t_2) = \overline{y_1(t_1)\, y_1(t_2)} = P_c \operatorname{Re} u(t_1-\tau)\, u^*(t_2-\tau)\, \rho(t_1-t_2) \times$$

$$\times e^{i\omega_0(t_1-t_2)} + (N_п + N_{01})\, \delta(t_1-t_2),$$

$$R_{12}(t_1, t_2) = R_{21}(t_1, t_2) = \overline{y_1(t_1)\, y_2(t_2)} = a N_п \delta(t_1-t_2),$$

$$R_{22}(t_1, t_2) = \overline{y_2(t_1)\, y_2(t_2)} = (a^2 N_п + N_{02})\, \delta(t_1-t_2),$$

where N_{01} and N_{02} are spectral densities of noises in channels;

a is the attenuation factor (taking into account the sign, depending on the number of the lobe) of interference in the second channel owing to the difference in the directivity factors of antennas.

Solution of equation (4.2.10) for this case can be presented in the form

$$V_{ik}(t_1, t_2) = B_{ik}(t_1, t_2) \operatorname{Re} u(t_1-\tau)\, u^*(t_2-\tau)\, e^{i\omega_0(t_1-t_2)}, \qquad (4.12.2)$$

In the case of slow fluctuation, coefficients $B_{jk}(t_1, t_2)$ do not depend on time and with accuracy up to immaterial factor A are equal to

$$B_{11}=A;\ B_{12}=B_{21}=-A\,\frac{\alpha N_п}{\alpha^2 N_п+N_{02}};\ B_{22}=A\left(\frac{\alpha N_п}{\alpha^2 N_п+N_{02}}\right)^2. \qquad (4.12.3)$$

Substituting these expressions into (4.2.9), we find

$$L_0(y)\sim\left|\int_0^T\left[y_1(t)-\frac{\alpha N_п}{\alpha^2 N_п+N_{02}}\,y_2(t)\right]u(t-\tau)\,e^{i\omega_0 t}\,dt\right|^2 \qquad (4.12.4)$$

Thus, optimum processing in the considered case includes subtraction (or addition if $\alpha<0$) of output of channels, and in the remaining does not differ from optimum processing of signal in one channel on a background of noises. If the directivity factor of an additional antenna directed toward interference coincides with the level of the corresponding parasitic lobe of the main antenna, and noise in the additional channel is small as compared with interference, then the factor during $y_2(t)$ can be replaced by one in (4.12.4).

During fast fluctuation of signal, the solution obtained is fully analogous. The difference of output voltages of channels must in this case pass through a filter with frequency response (4.3.8), be detected and be integrated during the time of observation and with this parameter h in (4.3.8) in this case is equal to

$$h=\frac{P_c}{2(N_п+N_{01})\,\Delta f_э}\cdot\frac{1}{1-\dfrac{(\alpha N_п)^2}{(N_п+N_{01})\,(N_п\alpha^2+N_{02})}}. \qquad (4.12.5)$$

Detection characteristics for the considered system of detection, as is easy to see, have the same form as for a single-channel system. Only the signal-to-interference ratio changes:

$$q_{эк}=\frac{P_c T}{2\left[N_{01}+N_{02}\,\dfrac{1}{\alpha^2+\dfrac{N_{02}}{N_п}}\right]}. \qquad (4.12.6)$$

From (4.12.6) it follows that when $\frac{N_{02}}{N_п}\ll 1$ the effect of interference during optimum processing is completely removed, and only the sum of noises in both channels affects the quality of detection. In practice, the effectiveness of compensation

turns out to be significantly lower because of inaccurate matching of amplification factors and values of directivity factors in channels.

It is necessary to note that compensation of interference can be carried out by such a method only when interference acts in the direction of the parasitic lobe of the main antenna. During a hit of target and interference in a beam of the diagram, levels of signals received on both channels become commensurable [taking into account the factor during $y_2(t)$ in (4.12.4)] and the signal is compensated simultaneously with the interference. However, even in this case the considered method has an important advantage: capture of a source of interference by angles becomes impossible when it hits the parasitic lobe of the antenna radiation pattern.

It is necessary to note that the given method of selection of signal may be, obviously, used with success for protection also from active interferences of another form, effective in the direction of the parasitic lobe. It is essential that a change in the intensity of interference, in time does not lower the effectiveness of the method.

4.12.2. Interference—Stationary Random Process

Above it was assumed that interference affecting the system of detection, has a spectrum width significantly larger than the spectrum width of modulation. Cases are encountered where this assumption is not fulfilled (narrow-band spot jamming, random pulse jamming, etc.). Examination of these cases not only allows us to estimate the effectiveness of such interferences, but also answers the question concerning the expediency of expanding the modulation spectrum from the viewpoint of protection from active interferences.

Thus, let us consider interference in the form of an arbitrary stationary random process, assuming that the spectrum width significantly exceeds the narrow-band filter transmission band in the receiver. This will allow us to consider the process at output of filter to be Gaussian, not being interested in the exact

distributive law of the interference, and to replace interference at input of filter by equivalent noise. With this, all detection characteristics can be used, which relate to a case of detection on a background of noise, in which the signal-to-noise ratio should be replaced by the signal-to-interference ratio. Equivalent spectral density of the interference at input of the narrow-band filter is determined, as it is easy to show, by relationship

$$N_{\text{экв}}=\lim_{T\to\infty}\frac{1}{2T}\int_0^T\int_0^T u(t_1)u^*(t_2)e^{i\Delta\omega(t_1-t_2)}r(t_1-t_2)\,dt_1dt_2, \tag{4.12.7}$$

where $\Delta\omega$ is the difference in carrier frequencies of the interference and the expected signal;

$r(t_1-t_2)$ is the envelope of the correlation function of interference.

Expressing $r(t_1-t_2)$ in (4.12.7) through spectral density of interference $S_{\text{п}}(\omega)$, we obtain

$$N_{\text{экв}}=\frac{1}{4\pi}\int_{-\infty}^{\infty}S_{\text{п}}(\omega)S_{\text{м}}(\omega-\Delta\omega)\,d\omega. \tag{4.12.7'}$$

Substituting this expression into the formula for the signal-to-noise ratio, we obtain a signal-to-interference ratio in the form

$$q_{\text{эк}}=\frac{q_0}{1+\frac{1}{4\pi N_0}\int_{-\infty}^{\infty}S_{\text{п}}(\omega)S_{\text{м}}(\omega-\Delta\omega)\,d\omega}. \tag{4.12.8}$$

When using formula (4.12.8), we should pay attention to the fact that $S_{\text{п}}(\omega)$ is the spectral density of a low-frequency envelope of interference, connected with the spectral density of interference $S'_{\text{п}}(\omega)$ by relationship

$$S'_{\text{п}}(\omega)=\frac{1}{2}[S_{\text{п}}(\omega-\omega_0)+S_{\text{п}}(\omega+\omega_0)].$$

The maximum of $S_{\text{п}}(\omega)$ is twice larger than the maximum of $S'_{\text{п}}(\omega)$, connected with which is the presence of 1/2 before the integral in (4.12.8).

If modulation is periodic, then

$$N_{\text{экв}}=\frac{\Omega_r}{4\pi}\sum_{k=-\infty}^{\infty}S_{\text{м0}}(k\Omega_r)S_{\text{п}}(k\Omega_r-\Delta\omega_x). \tag{4.12.9}$$

from which it is clear that in a narrow band of interference, smaller than the repetition frequency, it affects the receiver only when $\Delta\omega \approx k\Omega_r$. With this, due to the broadband nature of the modulation,

$$N_{\text{экв}} \approx \frac{\Omega_r}{4\pi} S_{\text{м}0}(\Delta\omega) \sum_{-\infty}^{\infty} S_{\text{п}}(\Delta\omega - k\Omega_r). \tag{4.12.10}$$

In the case of a band of interference, great as compared with the repetition frequency, it is possible to replace the sum in (4.12.9) by the integral

$$N_{\text{экв}} \approx \frac{1}{4\pi} \int_{-\infty}^{\infty} S_{\text{м}0}(\omega) S_{\text{п}}(\omega - \Delta\omega)\, d\omega. \tag{4.12.11}$$

In order to imagine the character of the dependence of the signal-to-interference ratio on the width of the spectra of interference $S_{\text{п}}(\omega)$ and modulation $S_{\text{м}0}(\omega)$, we approximate these spectra, for example, by П-shaped curves with a width of $\Delta f_{\text{п}}$ and $\Delta f_{\text{м}}$ respectively. Then, taking into account (4.12.11) and disregarding shift $\Delta\omega$, we obtain when $\Delta f_{\text{п}} < \Delta f_{\text{м}}$

$$q_1 \approx \frac{q_0}{1 + \frac{P_{\text{п}}}{2N_0 \Delta f_{\text{м}}}} = \frac{q_0}{1 + \frac{N_{\text{п}}}{N_0} \frac{\Delta f_{\text{п}}}{\Delta f_{\text{м}}}}, \tag{4.12.12}$$

where $P_{\text{п}}$ is power;

$N_{\text{п}}$ is the maximum of the spectral density of interference.

From (4.12.12) it is clear that the effectiveness of interference is determined by the relative magnitude of spectral density of equivalent noise $\frac{P_{\text{п}}}{2\Delta f_{\text{м}}}$, which diminishes during expansion of the spectrum of modulation. With this, by $P_{\text{п}}$ one should understand the power of the interference in the modulation band. Hence expansion of the modulation spectrum leads to an increase of noise-resistance, when the width of this spectrum is more than the width of the interference spectrum.

During expansion of the interference spectrum the following pattern is observed. At first, interference affects the system only during $\Delta\omega$, near to the magnitude of the multiple frequency of repetition, where effectiveness is lowered with expansion of the modulation spectrum. If the width of the prior interval of Doppler frequencies is significantly less than the repetition frequency, then it is rather

difficult for such interference to hit in this interval.

Narrow-band interference can be used as a false signal for withdrawal of the tracking system with respect to speed or false operation of the system of capture with respect to speed. To combat such effects it is possible, apparently, to use the immutability of the properties of interference depending upon the magnitude of delay of the expected signal: operations have to occur in the same channels, detuned with respect to speed, at all distances simultaneously.

When $\frac{\Omega_r}{2\pi} < \Delta f_п < \Delta f_м$ requirements for accuracy of adjustment of interference are considerably lowered. It is sufficient to have an accuracy comparable with the width of the modulation spectrum. The effectiveness of interference with this does not depend on the width of its spectrum and decreases during expansion of the modulation spectrum. When $\Delta f_п > \Delta f_м$ requirements for accuracy of adjustment of interference continue to lower. Effectiveness of interference, obviously, decreases during an increase of $\frac{\Delta f_п}{\Delta f_м}$ (an even greater part of the power does not affect the receiver) and is determined by the relative magnitude of spectral density of interference in the modulation band.

As an example of stationary interference, let us consider random sequence of pulses, whose moments of appearance are distributed evenly, and the number of pulses in a fixed time interval is distributed according to the law of Poisson. Phases of high-frequency filling of separate pulses will be considered random (incoherent sequence of pulses). With this, the correlation function of a sequence of n pulses, appearing at interval T, much larger than pulse duration, is equal to

$$R_п(\tau) = \operatorname{Re} P_п \frac{n}{T} \int_{-\infty}^{\infty} u_п(t)\, u_п^*(t-\tau)\, dt\, e^{i\omega_0\tau},$$

where $P_п$ is power;

$u_п(t)$ is complex modulation of pulse.

During $T \to \infty$ $\frac{n}{T} \to \nu$ is the average frequency of pulses.

Finally, for the spectral density of interference $S_{\text{п}}(\omega)$ we obtain

$$S_{\text{п}}(\omega)=P_{\text{ср}}S_{\text{мп}}(\omega), \tag{4.12.13}$$

where $P_{\text{ср}}$ is the average power of interference;

$S_{\text{мп}}$ is the spectral density of the modulation of interference;

$$S_{\text{мп}}=\frac{|U_{\text{и}}(\omega)|^2}{\tau_{\text{и}}};$$

$\tau_{\text{и}}$ is the effective pulse duration.

The effectiveness of random pulse jamming, as one may see from (4.12.13), is determined by the form and width of the spectral density of modulation and is identical to the effectiveness, for example, of noise interference with the same spectral density.

This affirmation is true, of course, only under the condition that we consider a receiver designed for the detection of signal in noise and not having a special means of protection from random pulse jamming of the type of mechanisms [59] which cut off the receiver during the effect of a powerful pulse. The use of such means is expedient, apparently, only with pulse signal. With continuous radiation, their presence would lead to a loss of part of the receivable signal which greatly lowers the effectiveness of the use of modulation during reception.

4.12.3. Relay Interference

Modern generators of relay interference have a very wide band (Ch. 1) and allow us to amplify and re-radiate signals with any practical law of modulation. Owing to motion of a target carrying a jamming transmitter, the re-radiated signal acquires shift with respect to frequency and delay the same as a reflected signal. If additional shifts with respect to frequency and delay are absent, then this signal is distinguished only by power from that reflected from a target carrying interference. The presence of such a re-radiated signal can prove dangerous only when a

relay is fixed on a false target used for camouflaging a true target. Usually in relays additional delay of the re-radiated signal is carried out and also shift of signal with respect to frequency can be carried out. In this case, relay interference can be used also to camouflage a target carrying interference.

To discern a signal from a target and interference, additional delay of re-radiated signal can be used. Out of a group of signals the nearest should be taken. If there are several targets or high ambiguity with respect to distance, then this method can be ineffective.

If the power of interference is many times more than the power of signal, then this difference in power, in principle, can also be used for selection of target. In particular, during a hit of powerful relay interference on the parasitic lobe (if sensitivity of relay allows such a possibility) interference compensation described in paragraph 4.12.1 can be used. In principle, discerning the signal and interference in this case can be, apparently, carried out, using the difference in statistical properties of these signals; however, such a method requires long tracking of both signals, and its possibilities as yet have not been studied.

4.13. Conclusion

In the process of analyzing systems of detecting coherent radar signals on a background of noises and of various kinds of active and passive interferences we have discussed basically three forms of problems having great practical value: the selection of the main signal, the selection of the processing method of the reflected signal and a quantitative appraisal of the quality of work of the various detection systems, which has been reduced to a calculation of the threshold signal-to-noise ratio or signal-to-interference ratio, corresponding to the given probabilities of correct detection and false alarm. These three forms of problems we will touch upon once again in this conclusion.

We will start with the methods of processing the received signal. As analysis has shown, optimum methods of processing depend considerably on the properties of the reflected signal and the form of interference affecting the radar. With the practical structure of radar it is natural, it seems to us, to take, for a basis, a processing method sufficiently near the optimum method of signal detection in noise, supplementing and partially modifying it during operation, depending upon the form of interference. With this, in the radar receiving mechanism, working without the participation of an operator, there should be an indicator determining, by the character of received signal, the form of interference and directing a change in the character of processing.

In a majority of cases it is possible to indicate a number of characteristics, by which the character of affecting interference can be determined sufficiently accurately. The distinctive peculiarity of noise interference, for example, is its broadband nature and the fact that it affects all receiving channels, detuned with respect to distance, equally. It is possible to include in the receiver an additional channel, tuned to distance, where the appearance of a target is excluded, and to use this channel as an indicator of noise interference. For an indication of passive interference in a large number of cases it is possible also to use its large expanse (as compared with the target).

For detection in noise in contemporary radar sets, using inherent coherent signal resolving power with respect to speed, it is necessary to use multi-channel detection systems. Separate channels should be detuned with respect to distance and speed to a magnitude of the order of the resolving power with respect to these parameters and should overlap prior ranges of distances and speeds of detectable targets. Certain elements of channels can be combined. This question was discussed in detail in Section 4.5. In every channel multiplication of the received signal by the expected signal should be produced (or gating of output of the reducing filter, which can be common for all channels), narrow-band filtration, detection,

and if the reflected signal fluctuates rapidly, incoherent accumulation during the time of observation.

In the presence of passive interferences, if interference is broadband ($\Delta f_{\text{п}} T \gg 1$) and the modulation spectrum drops sufficiently fast, the described system of detection on a background of noise can be used with success. In this case coherent accumulation carried out in a narrow-band filter ensures almost the same suppression of interference as the corresponding optimum processing.

In a case of narrow-band interference, for its suppression a system of period-by-period subtraction can be used. During the corresponding selection of multiplicity and the magnitude of attenuation in the delay circuits and during coherent accumulation of signal after subtraction, such a system, as analysis has shown, in a large number of cases is near optimum. In order to avoid changing the character of processing during the appearance of passive interferences, it is possible to use an increase in time for examining the directions occupied by interference. Such a method can give good results when the angular dimensions of interferences are small as compared with the width of the sector of survey.

Use of a special method of processing a received signal [coherent compensation of interference (Section 4.9)] is inevitable in the case of point interference with an assumed distance from a target smaller than the resolution range with respect to distance. Processing of such form can be carried out by means of multiplication of the received signal by the support signal equal to the difference between the expected signal from the target and the signal from interference, taken with the corresponding factor, or by means of corresponding subtraction of values of the output voltage of the reducing filter.

Let us turn to the question concerning selection of the main signal. Here we can mention the number of utilized carrier frequencies, the law of modulation on each of the working frequencies and concerning the width of the modulation spectrum, determining resolving power with respect to distance.

As analysis has shown, the simultaneous use of several working frequencies, spaced from each other far enough that the corresponding reflected signals are statistically independent, allows us to increase considerably the free-space range of a slowly fluctuating target and during the corresponding selection of frequencies to ensure reliable selection on a background of passive interference in a given speed range. If the signal from the target fluctuates rapidly, then the range, with an increase in the number of working frequencies, no longer increases, and from this point of view multi-frequency work becomes inexpedient.

The law of modualtion of the main signal from the viewpoint of the problem of detecting a single target on a background of noise has no value. The effect of this law begins to show only when it is necessary to distinguish several signals, which can be signals from several targets, signals from a target and passive interference, or a signal from a target and active interference.

Results of Section 4.10 show that the best selection of a target with respect to speed is ensured during the use of a signal with line spectrum, and if it is desired, to combine this property with high resolving power with respect to distance, the number of spectral lines must be increased. Such properties of the main signal are easiest to obtain, using a periodic signal with intraperiodic modulation, ensuring good distance resolution (for example, a pulse signal or continuous signal with phase-code manipulation). To remove ambiguity in distance and speed, inherent to the periodic signal, in those cases, when this ambiguity cannot be avoided by selection of repetition frequency, it is possible to use a change of repetition frequency in the process of operation or a combination of several frequency channels with various repetition frequencies (paragraph 4.10.3).

Width of the spectrum of intraperiodic modulation determines the extent of the resolution range with respect to distance. As discussion in paragraph 4.10.4 has shown, from the viewpoint of reliability of target detection on a background of noises

and passive interferences, it is expedient to decrease the resolution range only as long as it does not become less than the dimensions of the target.

In certain radar stations it can be expedient to change resolving power with respect to distance in the process of operation. An increase of resolving power after capture, with respect to angles and speed, can be used to decrease the number of channels in the system of detection (Section 4.5).

The comparative analysis of various forms of signals and the methods of their processing, the results of which were briefly enumerated above, is possible only with the use of their quantitative characteristics. In this chapter we obtained a large number of relationships determining the form of detection characteristics and allowing us to find the probability of correct detection and false alarm and the magnitude of the threshold signal-to-noise ratio or signal-to-interference ratio corresponding to the chosen probabilities. Many formulas for threshold signal-to-noise ratio (signal-to-interference) are sufficiently simple and can be used during engineering calculations.

It is necessary to note that during the analysis and synthesis of detection systems we have considered basically two extreme cases -- fast and slow fluctuation of reflected signal. For the intermediate case, the relationships obtained, were not defined due to essential calculating difficulties. Results for this case, relating to the synthesis of optimum systems, do not present, in our opinion, an essential practical interest (these systems are very complicated and will hardly be used, and moreover, they apparently give small gain as compared with the mechanisms usually utilized), whose detection characteristics it is impossible to mention.

The problem of determining detection characteristics in an intermediate case between fast and slow fluctuation very frequently appears in practice and the solution of it, furthermore, would allow us, finally, to establish where slow fluctuation ends and where fast fluctuation begins. On graphs illustrating the character of the dependence of threshold signal-to-noise (signal-to-interference) ratio on the

width of the spectrum of fluctuation of the signal, the corresponding sections of the curve were plotted by interpolation. For a whole number of problems the accuracy thus obtainable is apparently sufficient. Nonetheless, accurate calculation of detection characteristics on this section of change $\Delta f_c T$ at least for a particular case, presents essential interest and is an urgent problem.

It is possible to indicate in this chapter also several other questions relating to the detection of coherent signal not receiving a sufficiently complete reflection. In particular, there is the problem of synthesizing an optimum detection system designed for a group of interferences capable of acting jointly or separately. As already has been noted, such a system should possess properties of self-adjustment and include an indicator of interferences, directing a change in the character of signal processing and possibly a change in the properties of the main signal. This problem is very great and complicated. At present, we probably do not have even a sufficiently clear mathematical formulation for it.

Another problem is connected with the selection of the form of main signal. In Section 4.10 it was shown that a signal ensuring good target selection with respect to speed on a background of passive interferences should possess line spectrum. At the same time it is desirable to combine this property with high resolving power with respect to distance. A combination of these properties takes place in the periodic signal, which possesses, however, high ambiguity in distance and speed. Hence the question appears whether it is possible to form a signal with line spectrum and high resolving power with respect to distance, in which the deficiencies of periodic signal would be at least partially reduced. Solution of this problem would be, it seems to us, an important contribution to the theory of radar signals.

An important problem for investigation is also the problem of detecting a fluctuating coherent signal during variable observation time depending on the obtained realization and during the use of a multi-channel detection system. In our opinion, the solution of this problem is of special interest, taking into account the possible

presence of active and passive interferences.

And, finally, a whole number of problems appears in connection with optimizing survey and investigation. Enumeration of these problems has already been given in Chapter 3 (Sections 3.8 and 3.9). We will note only that the particular questions examined in Section 4.7, relating to this problem, indicate promising investigation in this field.

CHAPTER 5

DETECTION OF INCOHERENT PULSE SIGNAL

5.1. Introductory Remarks

In accordance with the definition introduced in Chapter 1, an incoherent signal differs from coherent by the presence of additional random phase shifts for separate periods of modulation. These phase shifts are stipulated by the method of generating of signals of such form. A high-frequency generator is started in every period by the pulse of the modulator. Here, the phase of oscillations of the generator changes from pulse to pulse in random form. Initial phases of oscillations is in seperate periods are independent and are distributed evenly in the interval $(0; 2\pi)$.

Obviously, the distinction between incoherent and coherent signals is kept only as long as the values of initial phases are not remembered and are not used during reception. In systems of selection of moving targets sometimes are used [65] so-called coherent heterodynes, phased by every pulse of the transmitter. Here, the signal on the output of the mixer is coherent.

The distinction between coherent and incoherent signals becomes immaterial in the case when the bandwidth of the filter, carrying out coherent storage, is significantly larger than the frequency of repetition.

If the off-duty factor of pulses is high, then such expansion of the band does not lead to distortion of modulation: for the pulse duration, the filter, as

before, works as an integrator. During the time between pulses, voltage on the output of the filter is decreased almost to zero, so that the neighboring pulses are detected independently. This circumstance allows us to use, in this chapter, in the investigation of the characteristics of detection of an incoherent signal, a number of results, received in Chapter 4 for a coherent signal.

The consideration in this chapter starts from the investigation of properties of the relation of verisimilitude for an incoherent signal. It seems that the form can be added to the optimum operations, acceptable from a technical point of view, only for extreme cases of slow and fast fluctuating at a very large and minute signal-to-noise ratio. To find the characteristics of detections, corresponding to optimum operations, in the general case is not possible. Therefore, proceeding from a number of other works [9, 19], we consider these characteristics only for a system, carrying out summation of squares of envelope pulses during the time of observation. Besides this system of detection, in the chapter is analyzed a system with binary storage, and also a system with integration of scanning by distance. For the considered systems, equally with reliability of detection on a background of noises is investigated the noise-resistance in reference to active interferences.

5.2. Relation of Verisimilitude For an Incoherent Signal

The character of optimum processing of an incoherent signal is determined by the relation of verisimilitude. This relation may be, obviously, found from the relation of verisimilitude for a coherent signal by means of averaging by additional phase shifts in every period. If the signal was coherent and additional phase shifts are absent, then the relation of verisimilitude for the case of detection of such a signal in a noise would be recorded in the form of (4.11.5):

$$\Lambda \sim \exp\left\{\sum_{l,k=1}^{n} v(l,k) f_l f_k^*\right\},$$

where

$$f_l = \int_{(l-1)T_r}^{lT_r} y(t)\, u(t-\tau)\, e^{i\omega t}\, dt; \qquad (5.2.1)$$

v(j, k) is determined by equation

$$v(j,\ k)+q\sum_{l}v(j,\ l)\rho(l,\ k)=q\rho(j,\ k); \qquad (5.2.2)$$

q--signal-to-noise ratio for the period.

In the presence in the j-th period of an additional phase shift θ_j in (4.11.5) f_j is replaced by f_j exp ($i\theta_j$). Averaging by all θ_j, we obtain

$$\Lambda\sim\frac{1}{(2\pi)^{n}}\int\limits_{0}^{2\pi}\cdots\int\limits_{0}^{2\pi}\exp\left\{\sum_{j,\ k}v(j,\ k)A_jA_ke^{i(\theta_j-\theta_k)}\right\}d\theta_1\ldots d\theta_n, \qquad (5.2.3)$$

where $A_j=|f_j|$ is the value of the envelope on the output of the system intraperiod processing (reducing filter or IFA, if intrapulse modulation is absent).

Integration in (5.2.3) cannot be produced without introduction of additional limitations on elements of matrix $\|v(j,\ k)\|$. In connection with this, let us immediately turn to consideration of particular cases.

Let us assume that the signal-to-noise ratio during the time of observation nq is small as compared to one. Here, the exponent in (5.2.3), the index of which with high probability turns out to be minute, can be replaced by the first two members of its Taylor series. With accuracy up to members on the order of q^2, we obtain

$$\Lambda\sim 1+\sum_{j=1}^{n}v(j,\ j)A_j^2 \qquad (5.2.4)$$

Optimum interperiod processing consists, in this case, of summation of squares of envelope signals, occuring in various periods on the output of the system intraperiod processing, with coefficients v(j, j), which in extreme cases of fast and slow fluctuations of a signal do not depend on j. Processing of such form can be carried out with the help of a square-law detector and storing mechanism (for example, a potentialoscope).

The same result occurs, as is simple to see, with an arbitrary signal-to-noise ratio, if the correlation of fluctuations of signal in neighboring periods is absent.

Here, the relation of verisimilitude has the form

$$\Lambda\sim\exp\left\{\frac{q}{1+q}\sum_{j=1}^{n}A_j^2\right\}.$$

With a large signal-to-noise ratio q elements of matrix $\|v_{jk}\|$ have, relative to q, a magnitude on the order of one. If in the received signal y(t) there is a

useful signal (exactly in this case, it is desirable to continuously reproduce the relation of verisimilitude on the output of the receiver), then with high probability, coefficients at $e^{i(\theta_j-\theta_k)}$ in (5.2.3) have a magnitude on the order of q and integrand in (5.2.3) quickly diminishes by measure of removal from maximum, taking place at $\theta_1=\theta_2\ldots=\theta_n$. In connection with this, the magnitude of the integral is determined mainly by the behavior of the function in the nearest environment of the maximum, where $e^{i(\theta_j-\theta_k)}$ it is possible to decompose into Taylor series by $\theta_j-\theta_k$. As a result, we obtain

$$A\sim\frac{\exp\left\{\sum\limits_{j,k}v(j,k)A_jA_k\right\}}{\left|\delta_{jk}\sum\limits_{l}v(j,l)A_jA_l-v(j,k)A_jA_k\right|^{1/2}}. \qquad (5.2.5)$$

The coefficient at the exponent depends on $A_1,\ldots,A_n$ significantly less, than the exponent, and its presence during interpretation and embodiment of optimum operations can not be considered. Basically, these operations reduce to the formation of a quadratic form $\sum\limits_{j,k}v_{jk}A_jA_k$. To a formation of analogous form is reduced also the optimum operations during coherent signal, only there the direct results of intraperiod processing, and not their moduli are entered in it. As was indicated in the examination of a coherent signal, the form of such kind can be received as the result of transmission of the converted signal through a storing filter, coordinated with the spectrum of fluctuations of the reflected signal (during fast fluctuations) or in the time of observation (during slow fluctuations), quadratic detection and subsequent storage during the time of observation (the latter only in the case of processing of envelopes $A_1,\ldots A_n$ and quadratic detection is replaced by raising to a square, which during slow fluctuation, when subsequent storage is absent, is an inverse operation (see Chapter 3) and can be rejected. Thus, during slow fluctuation, the optimum interperiod processing for the considered case of $q \gg 1$ reduces to linear detection and storage. During fast fluctuation, this accumulation should be carried out with the help of a pulse filter, coordinated with the spectrum fluctuations, the result of storage should be raised to a square

and, in turn, be stored already for the whole time of observation. In a limited case of fast fluctuations, when the amplitudes of neighboring pulses are statistically independent, storage after the linear detector disappears and optimum processing reduces, as already was noted, to summation of squares of the envelopes.

Actually, in radar stations using an incoherent signal, the interperiod processing consists in detection and storage of a predetected signal during the time of observation. The characteristics of utilized detectors, as already was noted in Chapter 2, are similar to quadratic at small input signals and to linear with a high level of the signal. Thus, in existing radars, processing is used which is similar to optimum in two extreme cases of fast and slow fluctuations. In the interval between these cases, optimum processing considerably differs from utilized, begin significantly more complicated. Unfortunately, mathematical difficulties do not allow us to produce a sufficiently strict comparison of these methods of processing, which would permit us finally to solve the question about expediency of transition to the optimum method. The accumulated experience of comparison of various methods of signal processing allows us to assume that in this case the transition to optimum processing does not provide an essential gain. The effectiveness of both compared methods is ensured by a decrease of the influence of fluctuations during accumulation of independent values of the signal from the output of the detector and hardly considerably changes upon certain modification of the method of accumulation.

Regarding the dependence of threshold signal-to-noise ratio on the form of detector characteristic, then to this question, as it is known, at the dawn of the development of radar theory, was devoted a great deal of theoretical and experimental works, as the result of which, it was fixed that the influence of this characteristic is hardly essential. In the future, in the calculation of characteristics of detection of incoherent signal, we shall consider the detector to be quadratic, in order to receive final results in sufficiently simple form.

In the investigation of various systems of detection, we will be interested only in post-detector processing, considering intraperiod processing of the optimum type. Calculation of deviations from optimality can be done in this case, as it is easy to see, with the help of the same relationships as with the coherent signal (see Section 4.4).

5.3. Characteristics of a System with Summation of Squares of Envelopes

We shall calculate the characteristics of detection for a system, on the output of which is formed and compared with the threshold the magnitude

$$L' = \sum_{j=1}^{n} A_j^2 = \sum_{j=1}^{n} |f_j|^2. \tag{5.3.1}$$

It is easy to see that expression (5.3.1) is a partial case of formula (4.11.7), when $v(j, k) = \delta_{jk}$. Considering this circumstance and using (4.11.8)--(4.11.10), for the characteristic function of magnitude L' we obtain

$$\Psi(\eta) = \frac{1}{|\delta_{jk} - i\eta R_{jk}|} = \exp\left\{ \int_0^{i\eta} \sum_{j=1}^{n} G(j, j; \gamma)\, d\gamma \right\}, \tag{5.3.2}$$

where R_{jk} --function of interperiod correlation of received signal [see (4.11.4')], and $G(j, j; \gamma)$ is determined by equation

$$G(j, k; \gamma) - \gamma \sum_{l=1}^{n} G(j, l; \gamma) R_{lk} = R_{jk}. \tag{5.3.3}$$

If the input of the system of detection is influenced only by noises, then $R_{jk} = \delta_{jk}$, the determinant in denominator (5.3.2) becomes diagonal and easily is calculated

$$\Psi_0(\eta) = (1 - i\eta)^{-n}.$$

This characteristic function corresponds to chi-square distribution with 2n degrees of freedom. In accordance with this, for the probability of false alarm, there takes place formula (4.11.82).

In the presence of useful signal

$$R_{jk} = \delta_{jk} + q\rho(j, k),$$

where q is the signal-to-noise ratio on the output of the system of intraperiod processing.

At optimum intraperiod processing, q is equal to the ratio of average power of signal for the period of spectral density of noise. Dependence of q on various withdrawals from optimum processing was considered in detail in Chapter 4.

Calculation of probability of exceeding of threshold for that case in general form cannot be done. Therefore, we, as earlier, will distinguish cases of fast and slow fluctuations. During slow fluctuation ($\rho(j,k) = 1$ $j,k = 1,\ldots,n$) the determinant in (5.3.2) also can be calculated immediately. As a result, for characteristic function $\Psi(\eta)$, we obtain an expression, fully analogous to (4.4.20)

$$\Psi(\eta)=\frac{1}{[1-i\eta(1+nq)](1-i\eta)^{n-1}}, \tag{5.3.4}$$

whence for equation of characteristics of detection analgous to (4.4.21) we have at $F \ll 1 - D$

$$D\approx \exp\left\{-\frac{K_{2n}^{-1}(F)-2(n-1)}{2(1+nq)}\right\}. \tag{5.3.5}$$

From (5.3.5) for threshold signal-to-noise ratio q_0, corresponding to probabilities F and D, we obtain

$$q_0\approx\frac{K_{2n}^{-1}(F)-2(n-1)}{2\ln\frac{1}{D}}-1. \tag{5.3.6}$$

Comparison of (5.3.6) and (4.4.28) shows that dependence of threshold power of signal on the number of incoherently stored pulses carries the same character, as dependence of q_0 on the product of the transmission band of the filter for the time of observation during detection of coherent signal. Correspondingly, the loss, stipulated by use of incoherent accumulation instead of coherent, coincides with the loss, stipulated by expansion of the filter band as compared with that coordinated $\left(\Delta f_\phi\approx\frac{1}{T}\right)$ in a number of times, equal the number of stored pulses n. If n is great, then threshold q_0 owing to incoherent accumulation is increased by approximately

$$\sqrt{n}\,\frac{\Phi^{-1}(1-F)}{\ln\frac{1}{F}}\ \text{times.}$$

During fast fluctuation, the solution of equation (5.3.3) can be received by Fourier transform. Substituting this solution in (5.3.2), we obtain

$$\Psi(\eta)=\exp\left\{-\frac{n}{2\pi}\int_{-\infty}^{\infty}\ln\{1-i\eta[1+qS(\lambda)]\}\,d\lambda\right\}, \qquad (5.3.7)$$

where $S(\lambda)$ is spectral density, corresponding to the function of correlation $\rho(j,k)$.

Using (5.3.7), it is simple to find the semi-invariants of the sought distribution

$$\varkappa_\nu=(\nu-1)!\,\frac{n}{2\pi}\int_{-\infty}^{\infty}[1+qS(\lambda)]^\nu\,d\lambda. \qquad (5.3.8)$$

For calculation of characteristics of detection at large Δf_cT can be used normal approximation (4.4.10) or Edgeworth series (4.4.11). In a limited case of fast fluctuations, when $\Delta f_cT\gg n$ and the neighboring pulses fluctuate independently, a clear expression can be obtained for dependence $q_0(D,F,n)$, which coincides, obviously, with (4.6.3) (upon replacement of m by n), since in this case we also deal with incoherent accumulation of a certain number of independent components of a signal.

The Edgeworth series can be used for calculation of characteristics of detection also at arbitrary values of Δf_cT. In connection with this, it is useful to give here an expression for semi-ivariants, not using solution of equation (5.3.3) by the method of Fourier, received for the case of $\Delta f_cT\gg 1$.

We shall find the solution to (5.3.3) in the form of a series

$$G(j,k;\gamma)=\sum_{\nu=0}^{\infty}G_\nu(j,k)\gamma^\nu, \qquad (5.3.9)$$

by substituting which in (5.3.3) and equating the coefficients at identical degrees ν in both parts of the equality, it is simple to be convinced that

$$G_\nu(j,k)=\sum_{l=1}^{n}G_{\nu-1}(j,l)[\delta_{lk}+q\rho(l,k)], \qquad (5.3.10)$$

i.e., that matrix $\|G_\nu(j,k)\|$ is obtained by raising the matrix $\|\delta_{jk}+q\rho(j,k)\|$ to the $(\nu+1)$th degree. By subtituting (5.3.9) in (5.3.2), we obtain

$$\Psi(\eta)=\exp\left\{\sum_{\nu=1}^{\infty}\left[\sum_{j=1}^{n}G_{\nu-1}(j,j)\right]\frac{(i\eta)^\nu}{\nu}\right\}. \qquad (5.3.11)$$

whence it follows that the sought semi-ivariants which are, as we know, coefficients at $(i\eta)^\nu/\nu!$ in the decomposition of $\ln\Psi(\eta)$ in a series by degrees of $i\eta$, are determined by formula

$$\varkappa_\nu = (\nu-1)!\sum_{j=1}^{n} G_{\nu-1}(j,\ j) = (\nu-1)!\,\mathrm{Tr}\,\|\delta_{jk}+q\rho(j,\ k)\|^\nu, \tag{5.3.12}$$

where $\mathrm{Tr}\,\|a_{jk}\|^\nu$ designates the sign of the v-th power of matrix $\|a_{jk}\|$.

From the matrix theory, we know that

$$\mathrm{Tr}\,\|a_{jk}\|^\nu = \sum_{j=1}^{n}\gamma_j^\nu,$$

where $\gamma_j\ (j=1,\dots,n)$ are the characteristic numbers of matrix $\|a_{jk}\|$, determined by equation

$$|a_{jk}-\gamma\delta_{jk}|=0. \tag{5.3.13}$$

It is easy to see that characteristic numbers β_j of matrix $\|\delta_{jk}+q\rho(j,\ k)\|$ are connected with characteristic numbers α_j of matrix $\|\rho(j,\ k)\|$ by the relationship

$$\beta_j = 1+q\alpha_j,$$

so that

$$\varkappa_\nu = (\nu-1)!\sum_{j=1}^{n}(1+q\alpha_j)^\nu. \tag{5.3.14}$$

The problem of calculation of semi-ivariants, entering Edgeworth series, was reduced, thus, to determination of characteristic numbers of matrices $\|\rho_{jk}\|$, i.e., to solution of equation (5.3.12) for this matrix. In the majority of practical cases, the solution of this equation can be done only with use of computational engineering.

In Fig. 5.1 are represented graphically the results of calculation of characteristics of detection, conducted by I. N. Amiantov and Yu. G. Sosulin for signal with exponential function of correlation.

From the figure it is clear that dependence $q_0(n)$ at constant $\Delta f_c T>0$ has a minimum, i.e., there exists an optimum number of pulses, into which it is expedient to divide the power of the emitted signal, in order to ensure maximum range. Existence of the optimum is connected with the effect of decrease of relative magnitude of fluctuations in the use of several not completely correlated random signal components.

At $\Delta f_c T = 0$ an optimum, naturally, is absent and the threshold signal-to-noise ratio q_0 monotonically grows with increase of number of pulses, between which the emitted power is distributed.

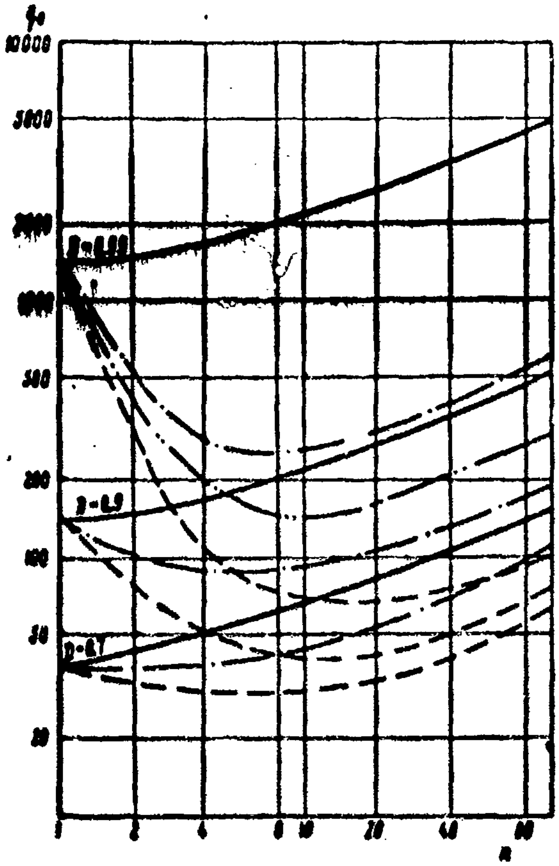

Fig. 5.1. Dependence $q_0(n, \Delta f_c T)$ for incoherent signal: —— $\Delta f_c T = 0$; —·—·— $\Delta f_c T = 0.5$; —··— $\Delta f_c T = 1$; — — — — $\Delta f_c T \gg n$.

This growth is connected with the decrease of power of signal, processed coherently (power of separate pulses). It is necessary to note that at $\Delta f_c T > 0$ at sufficiently large n there also takes place growth of q_0 with increase of n, whereby, as one may see from the figure, q_0 grows with approximately the same speed, as during slow fluctuation, i.e., approximately proportional to $\sqrt{n}$.

The proportionality factor, depending on probabilities of correct detection and false alarm and on the relationship between time of observation and width of spectrum of fluctuations for the case of fast fluctuations, can be found, considering the distributive law of stored signal to be Gaussian. The corresponding equation

of characteristics of detection has the form

$$D = 1 - \Phi\left[\frac{\Phi^{-1}(1-F) - q\sqrt{n}}{\sqrt{1 + 2q + \frac{q^2}{2\pi}\int\limits_{-\infty}^{\infty} S^2(\lambda)\, d\lambda}}\right]. \qquad (5.3.15)$$

At $\Delta f_c T \gg n$ the neighboring pulses fluctuate independently and $S^2(\lambda) \equiv 1$. Here,

$$q_0 \approx \sqrt{n}\, \frac{\Phi^{-1}(1-F) + \Phi^{-1}(D)}{1 - \frac{\Phi^{-1}(D)}{\sqrt{n}}}. \qquad (5.3.16)$$

If $n \gg 1$, the second component in the denominator can be disregarded and q_0 turns out to be proportional to $\sqrt{n}$. At very large n, when n turns out to be comparable with $\Delta f_c T$, the neighboring pulses become correlated. Here,

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} S^2(\lambda)\, d\lambda \approx \frac{an}{\Delta f_c T}, \qquad (5.3.17)$$

where a is the coefficient, depending on the form of spectrum of fluctuations. For the case of exponential function of correlation a = 0.5.

By substituting (5.3.17) in (5.3.15) and disregarding components 1+2q under the radical (this is possible to do, if $q_0 \gg 1$), we obtain

$$q_0 \approx \sqrt{n}\, \frac{\Phi^{-1}(1-F)}{1 - \sqrt{\frac{a}{\Delta f_c T}}\, \Phi^{-1}(D)}. \qquad (5.3.18)$$

Using Fig. 5.1, it is easy to check that calculation by approximate formulas (5.3.16) and (5.3.18) gives results fully satisfactory in accuracy in regions of large n.

From the curves in Fig. 5.1, it is clear that the dependence $q_0(\Delta f_c T)$ at constant n considerably differs from the analogous dependence at coherent signal.

With an incoherent signal, q_0 monotonically diminishes with the growth of $\Delta f_c T$, when with the coherent signal, this dependence had a very clearly expressed minimum. This distinction is connected with the fact that in the considered case, an increase of Δf_c does not lead to lowering of effectiveness of coherent processing, carried out only within the limits of each pulse, inasmuch as fluctuating changes of signal for pulse duration are assumed to be small. Due to this assumption, we

always remain in that region of $\Delta f_c T$, where the useful influence of expansion of the spectrum of fluctuations (increase of number of independent components of signal) acts in full measure, and the harmful influence (impairment of coherence) is completely absent, i.e., always remains on the left of the minimum.

5.4. Effectiveness of Use of Several Frequency Channels and Several Independent Scanning Cycles

The questions which are the subject of this paragraph, were already considered in the proceding chapter for the case of the coherent signal. It was shown that during slow fluctuation of a reflected signal, there exists an optimum number of statistically independent components of the signal, ensuring a maximum of free-space range with a given total average power. These components can exist either due to emission on several frequencies, or due to the application of several cycles of space scanning. With fast fluctuations, the threshold signal-to-noise ratio q_0 monotonically increases with the increase of the number of components, so that the best results are provided in this case by a single-frequency radar set or a radar set with slow scanning (one cycle for the entire time used in detection).

Analogous dependences with incoherent signal have a number of peculiarities, which we will now consider. We shall start from the case of multifrequency work, when the power of each emitted pulse is distributed equally between m frequency channels. If the frequency difference between channels is sufficiently great, then the corresponding reflected signals fluctuate independently. As was shown in Chapter 4, the optimum method of joint processing of such signals is the summation of results of processing of separate signals. The corresponding characteristic function $\Psi_m(\eta)$ for total signal is equal to the product of characteristic functions of results of processing of separate frequency components. We shall consider that in every frequency channel there is carried out summation of squares of envelopes of separate pulses. Here, the characteristic functions of output voltages of separate channels are determined by corresponding formulas in Section 5.3. In

particular, when the reflected signal fluctuates slowly, in accordance with (5.3.4)

$$\Psi_m(\eta)=\frac{1}{\left[1-i\eta\left(1+\frac{q_0}{m}\right)\right]^m(1-i\eta)^{m(n-1)}}, \tag{5.4.1}$$

where q_0 is the relation of total power of signal to spectral density of noise (one-sided).

If mn is sufficiently great, then the distributive law of total signal in the absence of a target can approximately be considered normal. Then, at $1-D \gg F$, we have

$$D\approx K_{2m}\left[\frac{2\Phi^{-1}(1-F)\sqrt{mn}}{1+\frac{q_0}{m}}\right], \tag{5.4.2}$$

whence

$$q_0\approx m\sqrt{mn}\,\frac{2\Phi^{-1}(1-F)}{K_{2m}^{-1}(D)}, \tag{5.4.3}$$

Dependence $q_0(m)$ at $F = 10^{-6}$, $n = 100$ and various D is shown in Fig. 5.2. At $D > 0.5$, function $q_0(m)$, as also in the case of coherent signal, has a minimum, which corresponds to the optimum number of independent components $m_{\text{опт}}$, somewhat less, than for a coherent signal. Function $q_0(m)$ quickly changes at small m, so that for obtaining a gain in distance, near to maximum, it is sufficient to use 2--4 frequency channels.

During fast fluctuation in a limit, when $\Delta f_e T \gg n$ and neighboring pulses fluctuate independently, the use of m frequency channels is equivalent to an increase of m times the number of pulses. Here, in accordance with (5.3.16) q_0 is increased by approximately $\sqrt{m}$ times. It follows from this that by measure of increase of $\Delta f_e T$, the optimum number of channels decreases. With n, comparable to $\Delta f_e T$, and $\Delta f_e T \gg 1$ analogous to (5.3.18), we have

$$q_0\approx\sqrt{mn}\,\frac{\Phi^{-1}(1-F)}{1-\Phi^{-1}(D)\sqrt{\frac{n}{m\Delta f_e T}}}. \tag{5.4.4}$$

In the case of multifrequency operation, it is technically more simple, apparently, to have a radar set, in which the frequency channels operate alternately, i.e., to use retuning of frequency. Here, the received signal is divided into m sections, fluctuating independently. Such a method, obviously, is completely

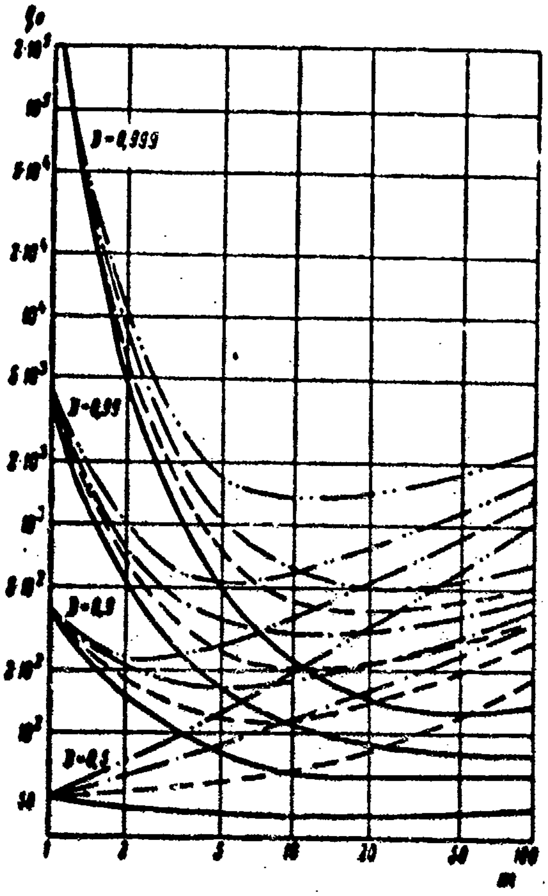

Fig. 5.2. Dependence of threshold signal-to-noise ratio on the number of independently fluctuating components with various methods of their obtainment and processing: —— – retuning of frequency (subdivision into cycles) during accumulation; - - - - subdivision into cycles during independent comparison with threshold; - . - . - - multifrequency operation during accumulation; - .. - .. - - multifrequency operation during independent comparison with threshold.

equivalent to the distribution of total time, used for irradiation of the given target, between m cycles.

During slow fluctuation, the corresponding characteristics of detection can be received from those just now considered by means of replacement of mn by $n_з$ (n_0 is the number of jointly processed pulses, n is the number of pulses of each frequency) and q_0/m by q_0, inasmuch as at each frequency the total power of the transmitter is emitted. In accordance with this, the threshold signal-to-noise ratio turns out to be for that case approximately $\sqrt{m}$ times less, than during simultaneous operation of frequency channels (Fig. 5.2), and monotonically diminishes with the growth of m. Best results are obtained

at m = n_0, when each pulse is emitted at its own frequency and fluctuates independently of those remaining. Physically, the gain as compared with the case of channels, operating simultaneously, is explained by the fact that in the use of retuning, the power of separate pulses, processed coherently is increased.

During fast fluctuation $(\Delta f_c T \gg 1)$, the received signal turns out to be consisting of several independently and randomly variable components due to the fluctuations themselves. Retuning of frequency or subdivision into cycles does not lead to increase of number of components as long as $\Delta f_c T_1$ $\left(T_1 = \frac{T}{m}\right.$ is the time of operation at each of the frequencies or exposure time of target for one cycle of scanning) remains significantly greater than one. In accordance with this, the magnitude of threshold signal-to-noise ratio does not change. Upon further increase of m, the ratio q_0 diminishes, also tending in limit to the value corresponding to the independently fluctuating pulses.

With the use of several independent cycles of scanning, due to possible displacements of the target during the cycle and from considerations of technical convenience, it is more expedient to make a comparison of the output signal with the threshold in each cycle, taking the solution of the presence of a target, if at least in k cycles from m there occurs exceeding of threshold. For simplicity, let us consider the case, when k = 1. Corresponding to this case, the characteristics of detection can be received from the characteristics of detection for one cycle by replacement of $\beta = 1 - D$ by $\sqrt[m]{\beta}$, F by $\frac{F}{m}$, n and q_0 by $\frac{n}{m}$ and $\frac{q_0}{m}$.

At $\frac{\Delta f_c T}{m} \ll 1$ in accordance with (5.3.6)

$$q_0 \approx m\left[\frac{K^{-1}_{2n/m}\left(\frac{F}{m}\right) - 2\left(\frac{n}{m} - 1\right)}{2\ln\frac{1}{1 - \sqrt[m]{\beta}}} - 1\right]. \qquad (5.4.5)$$

A graph of dependence (5.4.5) at F = 10^{-6}; n = 100 and various D also is shown in Fig. 5.2. At D > 0.5, the dependence $q_0(m)$ has minimum at certain $m_{\text{опт}}$. The threshold signal-to-noise ratio turns out to be in this case somewhat smaller, than during accumulation of output voltages of simultaneously operating frequency

channels, but significantly larger, than during accumulation of independent cycles of scanning (and upon retuning of frequency).

During fast fluctuation $(\Delta f_c T \gg 1)$, an increase of m leads, obviously, to an increase of threshold signal, since here near to optimum processing, the accumulation of statistically independent signal sections, due to fluctuations, is replaced by independent comparison of the results of accumulation in these sections with the threshold.

Independent comparison with the threshold of results of processing of separate statistically independent components of a signal can be used also in case of simultaneously operating channels spaced by frequency. Such processing, in particular, takes place during simultaneous irradiation of a detected target by several radar stations. During slow fluctuation, calculation of characteristics of detection can also be done on the basis of (5.3.6). Replacing β in (5.3.6) by $\sqrt[m]{\beta}$, F by $\frac{f}{m}$ and q_0 by $\frac{q_0}{m}$, we obtain

$$q_0 \approx m\left[\frac{K_{2\nu}^{-1}\left(\frac{F}{m}\right)-2(\nu-1)}{2\ln\frac{1}{1-\sqrt[m]{\beta}}}-1\right]. \qquad (5.4.6)$$

Dependence $q_0(m)$ for that case, also shown in Fig. 5.2, has a minimum for $D > 0.5$ at m significantly smaller, than during accumulation of output voltages of frequency channels. Curves of $q_0(m)$ pass significantly above the corresponding curves for other methods of processing of statistically independent signal components considered here.

5.5. Visual Detection

In a great number of radar stations, the solution on the presence of a target is taken by the operator on basis of observations of the output signal of the radar set on the screen of the cathode-ray indicator. The signal from the video detector, proceeding to the input of the electron-beam tube, is stored on the screen of tube, due to the inertness of the luminophor, covering the screen. Therefore, by the method of processing of the signal, the system of visual detection is similar to an automatic

system with incoherent accumulation of signal and, in particular, to a system with accumulation on a potentialoscope. The essential difference consists of the fact that the system of reading and relay are replaced during visual detection by the eyes and brain of the operator.

Any strict theory of visual detection is absent and will hardly appear quickly, since it is not possible to quantitatively consider the entire mass of psychological and physiological factors, introduced by the operator in the process of detection. The role of the theory in this question reduces to finding a method of calculation of characteristics of detection, the results of which are more or less well coordinated with experience. Comparison of results of calculation and experiment is devoted, in particular, to a significant part of the book "Threshold signals" [1], and also a great number of periodical articles [69-73].

Correct placement of the experiment for determination of the characteristics of visual detection is connected with a whole series of essential difficulties. For the most part, the experiment is placed in idealized conditions, whereby the validity of the received idealizations is determined only by the intuition of the experimenters. In particular, in the experiments, before the operator usually was placed the problem of taking a solution concerning in which of the limited number of fixed positions appeared a signal, if it is accurately known that it exists. Here, it remains indefinite such an important parameter, as the probability of false alarm, which significantly lowers the value of the received results.

For estimation of threshold signal-to-noise ratio in visual detection frequently is used the so-called "criterion of deviation": they consider that the signal can be detected with a probability, near to 0.5 if the difference of mean values of stored signal in points of the screen, where the useful signal is located and where it is not, is approximately equal to the mean-square deviation. This criterion gives result, quite near to experimental and, in particular, truly reflects the dependence characteristic for incoherent accumulation $q_s = nq \sim \sqrt{n}$.

For calculation of q_0 at various D and F it is necessary to obtain some kind of model of a system of visual detection. As such a model can serve, for example, a system with an accumulator in the form of a potentialoscope and relay. The operator, using observations only for one cycle of scanning, is not able, apparently, to ensure reliability of detection higher, than the proposed model, since conversions of the signal carried out by this model are near to optimum. If such a model is used, then for calculation of characteristics can be used the results of Section 5.3.

The results of the experiment, described in [71], show that in reality, the operator uses several cycles: with an increase of number of cycles, the probability of detection growed faster, than this would be during independent processing of cycles. It follows from this that an increase of range of the radar set with visual detection can be received by means of increase of speed of rotation of antenna. A reasonable limit of such increase should be fixed experimentally, since the mechanism of use of neighboring cycles is unknown to the operator. Conducted in the preceding paragraph, the calculation of characteristics of detection for various methods of processing of statistically independent components allows to assume that for visual detection, the range, near to maximum, can be received in 2--4 cycles during the time, used for detection of target.

5.6. Influence of Active Interferences On a System with Accumulation of Squares of Envelopes

We shall now investigate the influence of active interferences on a system of detection of incoherent signal with summation of squares of envelopes of separate pulses, subjected to intraperiod processing. As varieties of interferences, we shall consider noise and random pulse jamming. With respect to relay interference, we cannot add anything new to what has already been noted in the discussion of the coherent signal; and therefore, this form of interference is not specially considered here.

During the analysis of noise-resistance of incoherent systems we will not also

specially remain on the question of influence on these systems of passive interferences. The question of passive interferences was considered in detail in Section 4.11, where, along with optimum systems of detection, were considered systems with period-by-period subtraction of arbitrary multiplicity and subsequent incoherent summation. It is not difficult to see that, considering in final formulas the multiplicity to be equal to zero, we can obtain all necessary results for a system with incoherent accumulation. Analysis of these results immediately shows the low interference-stability of the considered system, which is fully coordinated with the conclusion made in Chapter 4 about the impossibility of effective protection from passive interferences without the use of selection by speed. In connection with this, a more detailed analysis of these results does not present practical interest.

Analysis of the influence of noise barrage interference on the system is conducted just as for the case of coherent signal. The distinction consists only in the fact that for normalization of interference due to the absence of a narrowband filter in the receiver, it is necessary to consider the width of the spectrum of interference to be large as compared to the width of the spectrum of modulation. In practice, this condition is usually fulfilled. This assumption allows to replace interference by equivalent white noise with corresponding spectral density $N_{\text{шп}}$. With automatic or manual gain control, ensuring constancy of frequency of false alarms, the harmful influence of interference reduces to a decrease of the equivalent signal-to-noise ratio. A corresponding decrease of the free-space range of the target with given probabilities D and F may be, as with the coherent signal, calculated by the formula (4.12.1).

For protection from interference, acting in the direction of the lateral lobe, as in the case of the coherent signal, the method of compensation of interference, described in paragraph 4.12.1 can be used.

Let us consider interference, represented by a random sequence of pulses with arbitrary intrapulse modulation $u_{\text{п}}(t)$. The influence of this interference on the

system of detection in the form considered here does not reduce to the influence of equivalent white noise. Strict analysis of this influence is connected with the calculation of interaction of the signal and interference in an amplifier of intermediate frequency (due to nonlinearity of the amplifier, which we cannot disregard due to the high level of interference) and video detector. In order to simplify calculations, we shall consider that due to limitation in the IFA, with coincidence of signal from the target and interference in time, the signal and noises of the receiver can be disregarded. Under this condition, the influence of interference on the system reduces to the appearance of an additional fluctuating component on the input and to vanishing of a certain part of pulses of the signal, coinciding with the interference.

If the frequency of tracking of pulses of interference is not very great, then vanishing of part of the pulses of the signal in the first order of approximation can be disregarded and considered only as an additional fluctuating component on the input of the relay, and also a change of amplification factor, stipulated by the influence of interference on the automatic gain control. With automatic gain control and sufficiently frequent interference such a case is possible, when the interference emerges from under the limitation. Usually this case corresponds to the impermissibly lowered reliability of detection and we will not consider it.

In accordance with the assumptions, the accumulated voltage in the presence of random pulse jamming (RPJ) is recorded in the form

$$V=\frac{1}{1+\frac{\overline{B^2}}{\overline{A^2_{\text{ш}}}}}\left(\sum_{j=1}^{n}A_j^2+\sum_{j=1}^{n}B_j^2\right), \qquad (5.6.1)$$

where B_j^2 is the square of the envelope of voltage of interference in the jth period on the output of the system of intraperiod processing,

$\overline{A^2_{\text{ш}}}$ is the mean value of the square of the envelope of noise on the output of the same system.

The distributive law of the first component in (5.6.1) in detail was investigated in the preceding paragraph. The distributive law of the second component at n $\gg$ 1

is possible to consider due to the independence of values of interference in various periods. We must find the mean value and dispersion of the second component and produce contraction of the distributive laws of components, in order to find distribution p(V).

Considering the appearance of pulses to subordinated to the law of Poisson with average frequency v, we have

$$\overline{\sum_j B_j^2} = n\frac{\nu E_{\text{п}}}{2}\int_{-\infty}^{\infty}|C_1(\tau,\ \Delta\omega)|^2\,d\tau = n\frac{E_{\text{п}}}{2}\nu\tau_1, \tag{5.6.2}$$

$$\overline{\left(\sum_j B_j^2\right)^2} - \overline{\left(\sum_j B_j^2\right)}^2 = n\frac{E_{\text{п}}^2}{4}\nu\int_{-\infty}^{\infty}|C_1(\tau,\ \Delta\omega)|^4\,d\tau = n\left(\frac{E_{\text{п}}}{2}\right)^2\nu\tau_2, \tag{5.6.3}$$

where $C_1(\tau,\ \Delta\omega)$ is the function of mutual correlation of pulses of interference $u_{\text{п}}(t)$ and signal u(t):

$$C_1(\tau,\ \Delta\omega) = \int_{-\infty}^{\infty} u_{\text{п}}(x+\tau)\,u^*(x)\,e^{i\Delta\omega x}\,dx, \tag{5.6.4}$$

and τ and $\Delta\omega$ is detuning of pulses of interference and expected signal from target in time and in frequency.

In (5.6.2)--(5.6.4) it is assumed that

$$\int_{-\infty}^{\infty}|u_{\text{п}}(t)|^2\,dt = \int_{-\infty}^{\infty}|u(t)|^2\,dt = 1. \tag{5.6.5}$$

$E_{\text{п}}$ in (5.6.2) and (5.6.3) designates the energy of pulse of interference on the input of the system of intraperiod processing, i.e., after passage through IFA. Obviously, in the presence of limitation

$$E_{\text{п}} \approx \frac{U_{\text{orp}}^2}{2}\tau_{\text{п}}, \tag{5.6.6}$$

where U_{orp} is the level of limitation,

$\tau_{\text{п}}$ is the effective pulse duration of interference.

Let us turn to the calculation of characteristics of detection. If there is no signal from the target, and if n $\gg$ 1, then the distributive law for V may be considered normal. Using (5.6.2), (5.6.3) and expressing the threshold of operation of relay by the probability of false alarm without RPJ, for the probability F_{rpj} of false alarm we obtain

$$F_{\text{rpj}} = 1 - \Phi[\gamma\Phi^{-1}(1-F)], \tag{5.6.7}$$

where

$$\gamma = \frac{1 + q_{\text{п}}\nu\tau_1}{\sqrt{1 + q_{\text{п}}^2\nu\tau_2}}, \tag{5.6.8}$$

and $q_{\text{п}} = \frac{E_{\text{п}}}{2N_0}$ is the interference-to-noise ratio in the period of repetition.

From (5.6.7) and (5.6.8) it follows that in the case RPJ, AGC does not ensure constancy of frequency of false alarms, which diminishes with an increase of frequency of interference.

We shall now calculate the probability of correct detection. For that, let us recall [see (5.3.4)], that during slow fluctuation the distributive law of stored signal is a contract of the exponential law and chi-square of distribution with 2(n--1) degrees of freedom. At $n \gg 1$, this law can be replaced by the normal one. Inasmuch as the distributive law of accumulated interference also is assumed to be normal, the law of p(V) is obtained as a result of contraction of exponential and normal distributions. Integrating p(V) from c to ∞ and replacing the integral of probability by the step function, we obtain

$$D_{\text{rpj}} \approx \exp\left\{-\frac{\Phi^{-1}(1 - F_{\text{rpj}})}{1 + q_0}\right\} = D^{\gamma}, \tag{5.6.9}$$

where D is the probability of correct detection with RPJ, and γ is determined by formula (5.6.8).

During fast fluctuation, assuming distribution for the stored signal in all cases to be normal, we have

$$D_{\text{rpj}} = 1 - \Phi\left[\frac{(1 + q_{\text{п}}\nu\tau_1)\Phi^{-1}(1 - F) - q\sqrt{n}}{\sqrt{1 + q_{\text{п}}^2\nu\tau_2 + 2q + q^2\frac{1}{2\pi}\int_{-\infty}^{\infty} S^2(\lambda)\,d\lambda}}\right]. \tag{5.6.10}$$

From (5.6.9) and (5.6.10), it is clear that D_{rpj} diminishes with an increase of frequency of interference.

The critical signal-to-interference ratio, corresponding to the given probability of detection, is increased in the presence of interference approximately in proportion to the average power of interference on the output of the system of intraperiod processing ($q_0 \sim 1 + \nu\tau_1 q_{\text{п}}$). The same increase of q also takes place during noise interference, so that these forms of interferences are approximately equivalent.

For protection from RPJ with great pulse power it is possible to use cutoff of the receiver for the time of income of the pulse, large as compared with the pulse of the signal. In this case, the probabilities of false alarm and correct detection will decrease by measure of growth of frequency of interference, since part of the stored pulses of noise or signal with noise falls, due to coincidence with interference. The number of stored pulses diminishes on the average of $(1-p_1)$ times, where p_1 is the probability of pulse of signal and interference. In accordance with this, the threshold signal-to-interference ratio for the period increases $1/\sqrt{1-p_1}$ times. Probability p_1 is equal to approximately $\nu(\tau_c + \tau_n)$, where τ_c and τ_n are the durations of pulses of signal and interference.

For protection from RPJ, it is possible also to use ordinary circuits of protection from non-synchronous interferences (delay lines for a period with cascades of coincidence, et.). In the use of a delay line with a cascade of coincidence, to the output of the latter passes only that pulse, which corresponds to the pulse in that same point of the preceding period, due to which the number of pulses of interference, appearing in the intervals between pulses of the signal decreases times, and accordingly, the frequency of false alarm is lowered.

5.7. System with Integration of Range Scanning

Sometimes it is sufficient only to establish the fact of the presence of the target, not indicating the distance to it. As was noted in Chapter 3, close in effectiveness to the optimum system of detection in this case is the multichannel system, which compares with the threshold the relation of verisimilitude for various values of distance. In certain cases, such a system can be unjustly complicated. In this paragraph we shall consider several simple systems, which in principle can be used for the solution of this problem.

Probably, the simplest and most evident method of detection of incoherent pulse signal without determination of magnitude of delay is integration of scanning by distance, carried out through the output of the video detector (or video

amplifier). If the a priori interval of delays $\Delta\tau$ is less than the period of repetition, the videoamplifier or IFA is gated with a duration of $\Delta\tau$. The time constant of the integrator is determined by the duration of the bunch.

Considering integration during the time T to be ideal, the detector quadratic and the video amplifier linear, it is possible to register voltage on the input of the relay in the form

$$V_{\cdot}=\sum_{j=1}^{n}\int_{0}^{\Delta\tau}A_j^2(t)\,dt, \tag{5.7.1}$$

where $A_j(t)$ is the voltage envelope on the output of the system of intraperiod processing (on output of coordinated IFA, if intrapulse modulation is absent) t seconds after the beginning of the j-th period.

Voltage $A_j(t)$ has a spectrum, limited by the transmission band of the system of intraperiod processing, coinciding with the width of the spectrum of modulation, and in accordance with the theorem of V. A. Kotel'nikov, can be completely characterized by discrete values in points, distant from each other by $1/\Delta f_{м}$. Examining this circumstance and considering that in the gate $\Delta\tau$ there is a signal from one target, (5.7.1) can be rewritten in the form

$$V\sim\sum_{j=1}^{n}A_{cj}^2+\sum_{l=1}^{n(\Delta f_{м}\Delta\tau-1)}A_{шl}^2, \tag{5.7.2}$$

where A_{cj} is the signal envelope with noise in the jth period,

$A_{шl}$ is the noise envelope.

All $n(\Delta f_{м}\Delta\tau-1)$ of values $A_{шl}$ are statistically independent and possess identical properties. Therefore, for calculation of distribution p(V), the belonging of $A_{шl}$ to some period does not have meaning, which is also considered in the notation (5.7.2).

The distributive law of p(V) can be received by contraction of laws for the first and second components in (5.7.2). The distributive law of the first component was considered in detail in Section 5.3. The second component is subordinated to chi-square distribution with 2 $n(\Delta f_{м}\Delta\tau-1)$ degrees of freedom ($A^2_{шl}$ can be represented in the form of the sum of squares of sine and cosine of the noise components on

the input of the detector, which are independent).

In the case of slow fluctuations, in accordance with (5.3.4), for characteristic distribution function p(V) we have

$$\Psi(\eta)=\frac{1}{[1-i\eta(1+q_0)](1-i\eta)^{nk-1}}, \qquad (5.7.3)$$

where $k=\Delta f_{\text{м}}\Delta\tau$.

From (5.7.3), for an equation of the characteristics of detection analogous to (5.3.6) we have

$$q_0\approx\frac{K_{2nk}^{-1}(F)-2(nk-1)}{2\ln\frac{1}{D}}-1\approx\sqrt{nk}\,\frac{\Phi^{-1}(1-F)}{\ln\frac{1}{D}}, \qquad (5.7.4)$$

from which it is clear that due to integration of scanning, the threshold signal-to-noise ratio is increased by $\sqrt{k}=\sqrt{\Delta f_{\text{м}}\Delta\tau}$ times, at the same value of probability of false alarm.

If the frequency of false alarm is given, the probability of false alarm in a system with integration of scanning turns out to be k times less than the probability for one section in the multichannel system. With real values of k and F, calculation of this circumstance leads only to an insignificant deviation from dependence $q_0(k)$.

In the case of fast fluctuations with the use of normal approximation, the equation of characteristics of detection has the form

$$D=1-\Phi\left[\frac{\Phi^{-1}(1-F)-\sqrt{\frac{n}{k}}\,q}{\sqrt{1+2\frac{q}{k}+\frac{q^2}{k}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}S^2(\lambda)\,d\lambda}}\right]. \qquad (5.7.5)$$

At q ≫ 1, the second component under the radical may be disregarded and for calculation of q_0 we can use the characteristics in Section 5.3, by increasing the threshold value of q_0 by $\sqrt{k}$ times.

It is interesting to note that in a similar manner, on the number of sections in distance depends the threshold signal for the system of detection with search. In search, the number of pulses, stored in every section, decreases k times, due to which $q_0\sim\sqrt{k}$. Thus, the system with integration of scanning and the system with search turn out to be equivalent. The very same also takes place for coherent systems during slow fluctuation of reflected signal [see (4.4.26)].

The effect of integration of scanning in a certain interval of delays frequently takes place also in a multichannel system (and, in particular, during visual detection) due to the contracted band $\Delta f_в$ of the video amplifier or the insufficiently high resolving power of the electron-beam tube. In these cases, the interval is approximately equal to $1/\Delta f_в$ or $b/v_р$ (b is the diameter of a spot on the screen of the tube, $v_р$ is the speed of scanning). Calculation of the influence of the mentioned technical errors can be performed with the help of the formulas received here.

It is possible to expect that the influence of noise in a system with integration of scanning will decrease with the use of a nonlinear video amplifier, in which the detected signal is limited from below. Here,

$$V=\sum_{k=1}^{n}\int_{0}^{\Delta\tau} f[A_з^2(t)]\,dt\approx\sum_{k=1}^{n} f[A_{ck}^2]+\sum_{l=1}^{n(k-1)} f[A_{шl}^2], \tag{5.7.6}$$

where f(x) is equal to zero at x less than the level of limitation of a and equal to x - a at x $\geq$ a.

Calculation of the characteristics of detection for that case can be produced only in the case of fast fluctuations of signal and interference, using the normal approximation. We shall find an expression for the mean value and dispersion of magnitude V. Considering that A^2_k is subordinated to exponential distribution, for $\overline{V}$ we easily obtain (with accuracy up to nonessential constant factor)

$$\overline{V}=n\int_{0}^{\infty} xp_c(x+a)\,dx+n(k-1)\int_{0}^{\infty} xp_ш(x+a)\,dx=$$
$$=n\left[(1+q)e^{-\frac{a}{1+q}}+(k-1)e^{-a}\right]. \tag{5.7.7}$$

Just as simply is calculated dispersion of random variable V without a signal from the target.

$$\overline{V^2}-\overline{V}^2=nk(2-e^{-a})e^{-a}. \tag{5.7.8}$$

Using formulas (5.7.7) and (5.7.8) and considering the law of distribution for V to be normal, it is easy to receive the following equation for threshold signal-to-

noise ratio q, corresponding to a 50% probability of detection:

$$(1+q)e^{\frac{a}{2}\left(1-\frac{2}{1+q}\right)}-e^{-\frac{a}{2}}=q_1\sqrt{2-e^{-a}}, \qquad (5.7.9)$$

where q_1 is the signal-to-noise ratio, corresponding to D = 0.5 without limitation (a 0). When $q_1 < 1$, the value of a, ensuring minimum q, turns out to be equal to zero.

When $q_1 \gg 1$ and $q \gg 1$

$$q \approx q_1\sqrt{2-e^{-a}}\,e^{-\frac{a}{2}}, \qquad (5.7.10)$$

from which it is clear that q monotonically diminishes with increase of a. The same result is also given by graphical solution of the equation. It shows that the normal approximation is impossible to use for determination of optimum level of a. The fact that this level exists is clear from physical considerations: with unlimited increase of a even at the level of operation, equal to the level of limitation, the probability of detection should infinitely diminish. Here, the conditions of application of the normal approximation are executed poorly, since there occurs a number of components different than zero in (5.7.6). Apparently, the optimum of a has a magnitude of the order q and is increased by measure of increase of the number of pulses n. A corresponding gain in the threshold signal-to-noise ratio can be approximately calculated by the formula (5.7.10). The question about the accurate value of the optimum and magnitude of gain, due to [illegible] is the simplest of all, apparently, in every separate case to solve experimentally.

Noise-resistance of the system with integration of scanning in reference to noise and random pulse jamming is determined, as is simple to be convinced of, by the same relationships as in the case of a system with accumulation of squares of envelopes. Introduction of limitation from below in no way, obviously, affects the noise-resistances in reference to random pulse jamming, inasmuch as the level of limitation is less than the amplitude of pulses of the interference.

Approximately the same results will be obtained, obviously, if instead of the

constant component of detected voltage, released during integration of scanning, we use some harmonic of frequency of repetition, released with the help of a filter. The signal-to-noise ratio for these harmonics turns out to be, in the case of a coordinated filter of intraperiod processing, equal to the signal-to-noise ratio on the zero harmonic.

5.8. System of Detection with Binary Accumulation of Signal

For accumulation of a signal in large time intervals, analog integrators, on the strength of a number of deficiencies inherent to them, turn out to be not wholly useful. In order to carry out accumulation with the help of a digital device, it is necessary first of all to convert all detected voltage into a number. Here, inasmuch as the number of digits in a digital installation is limited, there inevitably takes place replacement of voltage by the nearest of possible numbers, i.e., quantization.

The simplest case of quantization is quantization by two levels of 0.1, which is carried out, for example, by means of feeding a video signal, selected in the appropriate way according to distance, on a relay with one steady state. The total number of standardized pulses obtained, during the time of observation $T = nT_r$ (n is the number of periods of duration T_r) is calculated and is compared to the threshold k, exceeding of which indicates the presence of a target. As showed the simulation [75], such a method of processing quantized signals ensures, along with detection, sufficiently accurate location of a bunch of pulses, reflected from the target.

The main problem connected with binary accumulation, is the comparison of it with the analog type. It is also interesting to note the question of in what degree are the operations, carried out in this case on the quantized pulses, near to optimum.

We shall constitute the relation of verisimilitude for quantized signals. For

that, let us consider at first a sequence of pulses with fixed amplitudes, which correspond to signal-to-noise ratio $q_1,\ldots,q_n$. For realization, consisting of v ones and n - v seroes, the relation of verisimilitude in this case is recorded in the form

$$\Lambda(y; q_1,\ldots,q_n)=\prod_{i\in\nu}\frac{p(q_i, a)}{q(0, a)}\prod_{i\in n-\nu}\frac{1-p(q_i, a)}{1-p(0, a)}. \tag{5.8.1}$$

where $p(q_i, a)$ is the probability of exceeding the threshold of quantization a of the signal envelope with noise at signal-to-noise ratio q_i.

The first product in (5.8.1) is taken for all periods, in which there appeared a one, and the second product--for all remaining periods.

The relation of verisimilitude for a fluctuating signal can be received by averaging (5.8.1) for all q_i $(i=1,\ldots,n)$

$$\Lambda(y)=\int_0^\infty\cdots\int_0^\infty p(q_1,\ldots,q_n)\Lambda(y; q_1,\ldots,q_n)dq_1\ldots dq_n. \tag{5.8.2}$$

In the case of slow fluctuations with probability one and relation of verisimilitude $q_1=q_2=\ldots=q_n$

$$\Lambda(y)=\int_0^\infty p(q_1)\left\{\frac{p(q_1, a)}{p(0, a)}\right\}^{\nu}\left\{\frac{1-p(q_1, a)}{1-p(0, a)}\right\}^{n-\nu}dq_1. \tag{5.8.3}$$

The relation of verisimilitude $\Lambda(y)$ is a monotonically increasing function of the number of ones v, comparison of which with the threshold, carried out during binary accumulation, is the optimum for this case of operation. An analogous result is obtained, if pulses of the signal fluctuate independently. Here,

$$\Lambda(y)=\left\{\frac{p_0(q, a)}{p(0, a)}\right\}^{\nu}\left\{\frac{1-p_0(q, a)}{1-p(0, a)}\right\}^{n-\nu}, \tag{5.8.4}$$

where

$$\left.\begin{aligned}p_0(q, a)&=\int_0^\infty p(q_1)p(q_1, a)dq_1,\\ q&=\int_0^\infty q_1p(q_1)dq_1.\end{aligned}\right\} \tag{5.8.5}$$

In a case, intermediate between very slow and very fast fluctuation, $\Lambda(y)$ depends not only on the number of ones, but also on their distribution in sequences. This result is not sudden. From qualitative considerations it is quite clear that at a sufficiently large signal-to-noise ratio q, the ones in the sequences should with great probability be arranged in groups, the duration of which is determined

by the time of correlation of fluctuations. Calculation of $\Lambda(y)$ for that intermediate case is connected with significant difficulties.

Calculation of characteristics of detection, corresponding to binary accumulation, comparatively simply is conducted for the case of independent pulses. Here, the number of ones is subordinated to binomial law and for probabilities of correct detection and false alarm, we have,

$$D=\sum_{\nu=k}^{n}\binom{n}{\nu}p_1^{\nu}(q,a)[1-p_1(q,a)]^{n-\nu}, \qquad (5.8.6)$$

$$F=\sum_{\nu=k}^{n}\binom{n}{\nu}p^{\nu}(0,a)[1-p(0,a)]^{n-\nu}. \qquad (5.8.7)$$

Excluding k from here, we can find dependence D(F, q, a) and, maximizing D (minimizing q), by a—optimum magnitude of threshold of quantisation and corresponding value of k. The reverse can be done: express a by F and find the optimum value of k. The necessary calculations can only be done numerically.

Dependence q_0 (F, D, k) for independently fluctuating pulses already was considered in Chapter 4 in connection with the problem of survey. In Fig. 4.18, dependence q_0 (k) is shown graphically for various D and $F = 10^{-6}$, n = 10. From the graph, it is clear that dependence q_0 (k) is near optimum, lying in the vicinity of 3—5, and is rather weak. The optimum value of p(0, a) is $2\cdot10^{-3} \div 2\cdot10^{-2}$. Loss as compared with analog accumulation, as shows the comparison of corresponding curves in Fig. 4.18 and 5.1, is 1.5—1.7 db.

In the case, when $n \gg 1$ and $\Delta f_c T \gg 1$, it is possible to use, for calculation of characteristics of detection, the normal approximation. The mean value and dispersion of accumulated voltage are determined by formulas

$$\varkappa_1 = np_1(q,a);$$

$$\varkappa_2=\sum_{j,k=1}^{n}\int_0^{\infty}\int_0^{\infty}p_{jk}(r_1,r_2)\,dr_1\,dr_2-n^2p_1(q,a), \qquad (5.8.8)$$

where $p_{jk}(r_1,r_2)$ is the joint distribution of squares of envelopes of the fluctuating signal with noise in j-th and k-th periods [see (2.4.44)]

We shall use decomposition of $p_{jk}(r_1, r_2)$ in series according to Laguerre polynomials

$$p_{jk}(r_1, r_2)=\frac{1}{4\sigma^4(1+q)^2}\times$$
$$\times\sum_{\nu=0}^{\infty}R^{2\nu}(j-k)\left[\frac{1}{\nu!}\frac{d^\nu}{dr_1^\nu}r_1^\nu e^{-\frac{r_1}{2\sigma^2(1+q)}}\right]\times \tag{5.8.9}$$
$$\times\left[\frac{1}{\nu!}\frac{d^\nu}{dr_2^\nu}r_2^\nu e^{-\frac{r_2}{2\sigma^2(1+q)}}\right],$$

where σ^2 is the power of noise on output of the detector,

$$R(j-k)=\frac{\delta_{jk}+q\rho(j-k)}{1+q}$$

is the coefficient of interperiod correlation of totality of signal and noise on input of detector,

q is the signal-to-noise ratio,

$\rho(j-k)$ is the coefficient of correlation of signal fluctuations.

We shall assume that

$$\rho(l)=e^{-2\Delta f_c T_r |l|}. \tag{5.8.10}$$

By substituting (5.8.9), (5.8.10) in (5.8.8), we obtain

$$\varkappa_1 = n e^{-\frac{c}{2\sigma^2(1+q)}}, \tag{5.8.11}$$

$$\varkappa_2 = n e^{-\frac{c}{2\sigma^2(1+q)}}\left(1-e^{-\frac{c}{2\sigma^2(1+q)}}\right)+$$
$$+2n\sum_{\nu=1}^{\infty}\left(\frac{q}{1+q}\right)^{2\nu}\frac{1}{e^{4\nu\Delta f_c T_r}-1}\left[\frac{1}{\nu!}\frac{d^{\nu-1}}{dx^{\nu-1}}x^\nu e^{-x}\right]^2_{x=\frac{c}{2\sigma^2(1+q)}}. \tag{5.8.12}$$

The series (5.8.12) converges quickly and can be used in practical calculations. The results of such calculation are represented in Fig. 5.3 in the form of dependence of $\sqrt[4]{\frac{q_{2n}}{q_{1n}}}$ on n when D = 0.9, $\Delta f_c T = 10$. As can be seen from the graph, the loss, caused by replacement of analog accumulation by binary, is increased with the growth of n; however, the rate of this increase gradually diminishes.

Calculation of characteristics, corresponding to binary accumulation during slow fluctuation, may be omitted, averaging, by the signal-to-noise ratio, the characteristics of detection for a regular signal [see(5.8.6)]. The latter, quite in detail are considered in [74], where it was shown that the optimum in magnitude

of threshold of quantisation in the case of a regular signal also is hardly critical. Magnitude a is proposed to select by proceeding from the condition $p(0,a) \approx 0,1$. Here, the loss in the signal-to-noise ratio as compared with analog accumulation is 1.5 - 2 db. Averaging of characteristics, corresponding to the regular signal, in the magnitude of the signal-to-noise ratio can only be done numerically.

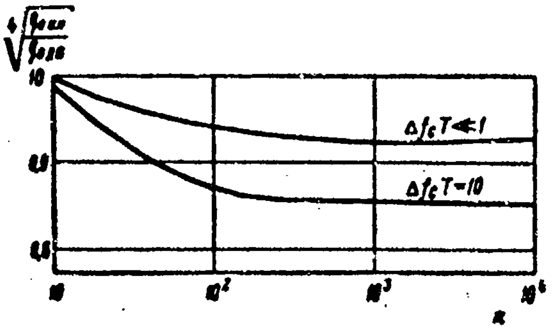

Fig. 5.3. Change of loss in distance due to replacement of analog accumulation by binary, depending upon number of stored pulses.

In Fig. 5.3 is shown the calculated dependence of $\sqrt[4]{\frac{q_{бн}}{q_{ан}}}$ on n when $\Delta f_c T \ll 1$ and D - 0.9. Comparison of curves of this figure shows that the loss of binary accumulation grows with the increase of $\Delta f_c T$.

The available results of comparison of systems of detection with binary and analog accumulation of detected pulses show that binary accumulation gives, in general, an unessential loss and can be, with success, used in cases, when realization of the system with analog accumulation encounters technical difficulties.

Binary summation may, obviously, be used also for accumulation of a detected signal in systems of detection of coherent signal. Here, quantisation should be carried out by amplitude and by time. The interval of quantisation in time should be small as compared with the time constant of the pre-detector narrow-band filter.

5.9. Conclusion

Comparison of coherent and incoherent systems of detection shows that incoherent systems frequently have significant loss in distance. In the most

typical case of slow fluctuations of target, this loss grows with the increase of number n of stored pulses approximately as $\sqrt[3]{n}$. Only in the case of independently fluctuating pulses of a signal when coherent joint processing of these pulses is impossible, there is no loss. Hence, follows the expediency of transition to the use of coherent systems of detection in those radar stations, for which such transition technically is possible.

In the solution of this question, one should consider that transition to coherent detection requires introduction, to the receiver of the radar set, usually of a very significant number of channels, detunned by Doppler frequency and covering a priori interval of frequencies (see Chapter 4). In incoherent systems, the multichannel effect in speed is usually absent, inasmuch as Doppler frequency turns out to be low as compared with the width of the spectrum of the pulse. In certain systems, the shown complication of the receiver turns out to be impermissible and for them incoherent signal processing is more acceptable.

With the incoherent signal, the system of detection should be multichannel with respect to distance, i.e., it should ensure independent accumulation and comparison with threshold of signals, arriving from various distances. Analysis of the system with integration of scanning, conducted in this chapter, showed that the abandonment of the multichannel effect entails an essential loss in the free-space ranges. For construction of a multichannel receiver storing devices can be used of the potentialescope type or delay lines with feedback, or blocks of gated amplifiers, the gates of which are detunned with respect to distance, with subsequent analog or binary accumulation. In the last case, accumulation can be carried out with the use of computers.

Loss with respect to distance, due to binary accumulation upon correct selection of threshold of the standardising device, turns out to be comparatively small (10 - 15%).

For decrease of number of channels with respect to distance, as with the

coherent signal, it is possible to use an increase of resolving power up to that required after detection, for example, by means of connecting intrapulse modulation.

During slow fluctuation of reflected signal, for increase of range simultaneous or alternate operation can be used on several carrier frequencies, and also, an increase of frequency of survey (subdivision of time, used for detection, in independently fluctuating cycles). For obtaining a gain in distance, near to maximum, in each of these cases it is usually sufficient to take 2 to 5 operating frequencies (cycles of survey). The biggest gain is obtained with the use of retuning of frequency with accumulation of results of processing separate pulses during the time of observation. For example, at 5 alternately utilized carrier frequencies, the free-space range with probability 0.9 is increased by 1.5 times.

Calculation of the possible presence of active interferences, as in the case of coherent signal, should, apparently, be produced with the help of additional devices introduced in the receiver for protection from interferences, connected upon appearance of any form of interference.

For protection from random pulse jamming, it is possible to apply selection of pulses by amplitude or by frequency of sequence. On the whole, the noise-resistance of incoherent systems of detection is significantly lower than coherent ones. This especially refers to passive interferences, for protection from which with an incoherent signal only the system with external coherence can be used, which was considered in the preceding chapter.

In spite of the abundance of incoherent systems of detection, some of the problems connected with these systems remain unsolved or are not solved until the end. In particular, there are no sufficiently full results on the characteristics of binary accumulations, concerning selection of thresholds of the standardizing cascade and relay and considering the presence and character of fluctuations of a useful signal. This problem obtained at present a large meaning in connection with the wide use of digital technology in systems of processing radar data. Its solution is connected with large calculating difficulties, the surmounting of which

will demand, apparently, the use of high speed computers.

Some interest is also represented by the investigation of characteristics of detection for optimum processing of an incoherent signal in those cases (see Section 5.2), when it does not coincide with accumulation of squares of envelopes. Apparently, such consideration is simple to make for a small number of stored pulses (for example, n = 2). Very desirable also is the stricter comparison of characteristics of detection with linear and square-law detectors. We enumerated typically "incoherent" problems. Furthermore, a number of unsolved problems, enumerated in the conclusion of Chapters 3 and 4, is common for coherent and incoherent signals. Here is the problem of optimum survey and target search, the problem of detection of nongaussian signal for the case, when it is impossible to disregard the problem of detection of a target on trajectories with the use of a memory from cycle to cycle, etc.

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